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The objective of this research was to develop a new technique for evaluation of the eigenvalues of the Graetz problem in slip-flow, a heat transfer problem for gases at low pressures or in extremely small geometries. In this investigation, the velocity distribution with slip-flow has been obtained and expressed simply in terms of the Knudsen number *Kn*. The expression shows that the velocity always increases as *Kn* increases. The relationship between *Kn* and molecular mean free path for a gas shows that *Kn* may become large enough to significantly affect the velocity distribution and consequently affect the heat transfer properties. A mathematical model of temperature distribution was established by combining the energy and momentum equations. A series solution was obtained by the method of Frobenius. Also, expressions for the local and overall Nusselt numbers as functions of *Kn* and the Graetz number were derived.

A new technique for evaluation of eigenvalues for the solution of the Graetz problem in slip-flow was developed. The computational results show that it is an effective method, and the lowest five eigenvalues were found for $0.02 \le Kn < 0.12$. For practical calculations, relationships between eigenvalues and *Kn* were obtained.

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EVALUATION OF THE EIGENVALUES OF THE GRAETZ PROBLEM IN SLIP-FLOW

TECHNICAL REPORT

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MARCH, 1996

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ABSTRACT

The objective of this research was to develop a new technique for evaluation of the eigenvalues of the Graetz problem in slip–flow — a heat transfer problem for gases at low pressures or in extremely small geometries. In this investigation, the velocity distribution with slip–flow has been obtained, expressed simply in terms of Knudsen (Kn) numbers. The expression shows that the velocity always increases as the Knudsen number increases. The relationship of Kn and molecular mean free path for a gas shows that Kn may become large enough to significantly affect the velocity distribution and consequently affect the heat transfer properties. A mathematical model of temperature distribution was established by combining the energy and momentum equations. A series solution was obtained by the method of Frobenius. Also, expressions for the local and overall Nusselt numbers were derived. All these expressions can be taken as functions of Knudsen numbers and Graetz numbers.

A new technique for evaluation of eigenvalues for the solution of the Graetz problem in slip–flow was developed. This method was based on the construction of a matrix. The computational results show that it is an effective method, and the lowest five values were found for Kn from 0.02 to 0.12. For practical calculations, relationships between eigenvalues and Knudsen numbers were obtained.

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Author Xining Wang

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NOMENCLATURE

A_w tube surface area [m^2]

a_k coefficient in Eq. (3.12)

$b_{i,k}$ coefficient in Eq. (4.8)

c characteristic velocity

c_a acoustic velocity

C_n coefficient in Eq. (1.3)

c_p unit heat capacity at constant pressure[J/kg–K]

c_v specific heat at constant volume [J/kg–K]

d_k coefficient in Eq. (1.7)

D tube diameter[m]

F specular reflection coefficient $(u_r - u_i) / (U_w - u_i)$

g magnification ratio in Eq. (5.1)

G $G(r^*)$: Graetz function

Gz Graetz number, $(RePr(D/L))$

h_x local convective heat transfer coef–ficient [W/m^2–K]

$\bar{h}_c$ average convective heat transfer

H characteristic dimension

coefficient [W/m^2–K]

k thermal conductivity [W/m–K]; number of terms in Equation (5.2)

Kn Knudsen number, (λ/D)

L length of tube[m]

$\overline{Nu}$ overall heat transfer coefficient, $(\bar{h}_c D/k)$

Nu_x local heat transfer coefficient, $(h_x D/k)$

p fluid pressure [Pa]

q heat flux per unit wall area [W/m^2]

Q heat transfer rate [W]

r radius[m]

r^* dimensionless radius, (r/R)

R tube radius[m]

Re Reynolds number, $(\rho u D/\mu)$

Re^* Reynolds number at c_a

R_g ideal gas constant [J/kg–K]

Pr Prandtl number, (ν/α)

T $T(r,x)$, temperature [K]

T_B bulk temperature [K]

T_L temperature at $x = L$ [K]

$(\Delta T)_{LN}$, log–mean–temperature

difference (LMTD) [K]

u velocity in x direction [m/s]

u_i average streamwise velocity of the incident molecules

u_r average streamwise velocity of the reflected molecules

U_w average streamwise velocity of the surface

v radial velocity

x distance along tube[m]

x^* dimensionless distance, (x/L)

Greek Symbols

α fluid thermal diffusivity, $(k/\varrho c)$ [m^2/s]

α_s thermal accommodation coefficient

β $(1+4\,Kn)$ coefficient in Eq. (4.1)

γ ratio of specific heats

Δ $\Delta = i-k$ in Eq.(4.8); $\Delta = (\overline{Nu} - Nu_x)$ in Fig. 5.9

η mean free path of gas

λ eigenvalue;

λ' eigenvalue divided by g

μ dynamic viscosity[kg/m s]

ν kinematic viscosity [m^2/s]

ϱ density[kg/m^3]

π 3.141592654

θ $(T-T_w)/(T_0-T_w)$ dimensionless temperature

θ_B $(T_B-T_w)/(T_0-T_w)$ dimensionless bulk temperature

$\theta_{B,L}$ dimensionless fluid bulk temperture at $x = L$

θ_{LN} dimensionless LMTD

Subscripts

0 at $x = 0$

B bulk

c centerline

m average

s slip–flow

w wall

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CHAPTER 1

INTRODUCTION

1.1 The Graetz Problem

By the end of the last century, the problem of forced convection heat transfer in a circular tube in laminar flow gained interest because of its fundamental importance in physical problems such as the analysis and design of heat exchangers.

The Graetz problem is a simplified case of the problem of forced convection heat transfer in a circular tube with laminar flow. With the assumptions of steady, incompressible and fully–established flow, constant fluid properties, no "swirl" component of velocity, a fully developed temperature profile, and negligible energy dissipation effects, Graetz (1883) originally solved this problem analytically. The solution by Graetz involved an infinite number of eigenvalues, and in his paper only the first two eigenvalues were evaluated.

Since the accuracy of the Graetz solution mainly depends on the number of eigenvalues, it is extremely important to obtain more eigenvalues, as Tribus and Klein (1953) pointed out. For seventy years, the research on this problem focused mainly on finding more eigenvalues. And Abramowitz (1953) employed a fairly rapidly converging series solution of the Graetz equation in making the calculation and found the lowest five values with much more accuracy. Sellars et al.(1956) extended the problem to include a more effective approximation technique for evaluation of the eigenvalues of the problem; they could get any number of eigenvalues as needed. This work solved the Graetz problem completely.

1.2 The Graetz Problem in Slip–Flow

Applications of microstructures such as micro heat exchangers have led to increased interest in convection heat transfer in micro geometries. Some experimental work has been done, such as the experimental investigations in microtubes (Choi et al.. 1991), in microchannels (Pfahler et al., 1991), and in micro heat pipes (Petersen et al., 1993). Therefore, appropriate models are needed to explain the significant departures in the micro–scale experimental results from the thermofluid correlations used for conventional–sized geometries. For example, the measured heat transfer coefficients in laminar flow in small tubes exhibits a Reynolds number dependence, in contrast to the conventional prediction for fully established laminar flow, in which the Nusselt number is constant (Choi et al., 1991). Also, an experimental investigation of fluid flow in extremely small channels showed that there are deviations between the Navier–Stokes predictions and the experimental observations (Pfahler et al., 1991).

Therefore, some effects and conditions that are normally neglected when considering macro–scale flow must be taken into consideration in micro–scale convection. One of these conditions is slip–flow (Flik et al., 1992, Beskok and Karniadakis, 1992). It has been found that the analytical model combined with slip flow conditions can fit the experimental data in microchannels with a uniform cross–sectional area (Arkilic et al., 1994) and with a non–uniform cross–sectional area (Liu et al., 1995).

Slip–flow occurs when gases are at low pressures or for flow in extremely small passages. At low pressures, with correspondingly low densities, the molecular mean free path becomes comparable with the body dimensions, and then the effect of the molecular structure becomes a factor in flow and heat transfer mechanisms(Eckert and Drake,1972).

The relative importance of effects due to the rarefaction of a gas can be indicated by the Knudsen number, a ratio of the magnitude of the mean free molecular path in the gas

to the characteristic dimension in the flow field. The effects of rarefaction phenomena on flow and heat transfer becomes important when the Knudsen number can no longer be neglected.

In defining when slip–flow occurs, Beskok and Karniadakis (1992) have proposed to classify four flow regimes for gases, as follows:

Continuum flow:	$Kn < 10^{-3}$
Slip–flow:	$10^{-3} < Kn < 0.1$
Transition flow:	$0.1 < Kn < 10$
Free molecular flow	$10 \leq Kn$

When slip–flow occurs, the gas adjacent to the surface, in contrast to its behavior in continuum flow, no longer reaches the velocity or temperature of the surface. The gas at the surface has a tangential velocity, and it slips along the surface. The temperature of the gas at the surface is finitely different from the temperature of the surface, and there is a jump in temperature between the surface and the adjacent gas. Eckert and Drake (1972) give expressions for the temperature jump condition and slip velocity at the surface. The slip velocity as a function of the velocity gradient near the wall can be expressed as follows:

$$u_s = -\eta \left(\frac{du}{dr} \right)_{r=R} \tag{1.1}$$

and Arkilic et al. (1994) give the expression as follows:

$$\frac{u_s}{c} = \frac{2-F}{F} \, Kn \left(\frac{du/c}{dr/H} \right)_{r=R} \tag{1.2a}$$

or

$$u_s = \frac{2-F}{F} \, \eta \left(\frac{du}{dr} \right)_{r=R} \tag{1.2b}$$

which includes the consideration of three accommodation coefficients represented by the specular reflection coefficient F. For most engineering surfaces, F has values near unity. In the case of F having a value of one, Eq. (1.2) becomes Eq. (1.1). For simplicity, in this investigation, Eq. (1.1) is applied to evaluate the velocity.

The original solution by Graetz (which was discussed above) is valid for continuum flow; however, for gases at low pressures or in extremely small tubes, the flow may enter the slip–flow regime, in which case the velocity at the tube surface is not zero. In this case, the heat transfer coefficient depends not only on the Reynolds number and Prandtl number, but also on the Knudsen number. This fact will no doubt make the model more complex and the evaluation of its eigenvalues more difficult. Therefore, a new technique is needed to evaluate the eigenvalues for a solution to the problem in slip–flow.

1.3 Related Research

Graetz (1883) originally solved the problem of forced convection heat transfer in a circular tube in laminar flow, with a developing temperature profile. Figure 1.1 shows the geometry and conditions for this problem.

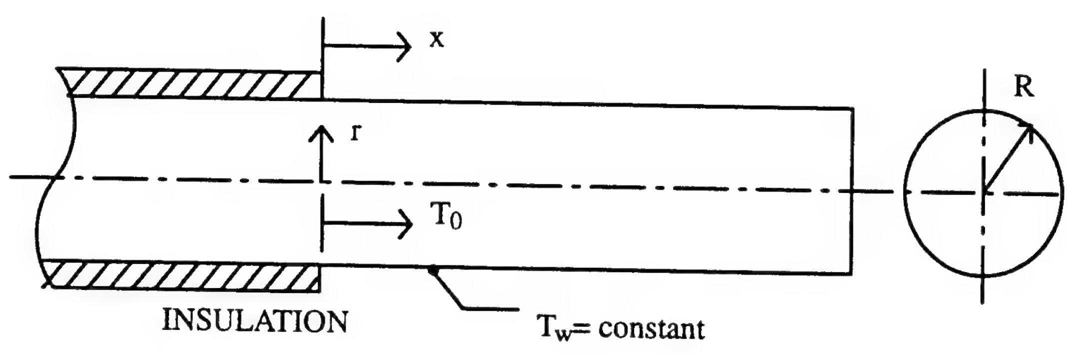

Fig. 1.1 Coordinate system and thermal boundary conditions

The mathematical statement of this problem is as follows:

$$T = T(x, r)$$

$$u\varrho C_p \frac{\partial T}{\partial x} = \frac{k}{r}\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r}) \tag{1.3}$$

$$u = 2u_m[1-(r/r_0)^2]$$

with the boundary conditions:

$T(R, x) = T_w$ for $x > 0$

$T(r, 0) = T_0$ for $x \le 0$

$T(0, 0) = T_0$ for $x \le 0$

The nondimensional form of the problem is:

$$\frac{\partial \theta}{\partial x^*} = \frac{1}{1-r^{*2}}\frac{1}{r^*}\frac{\partial}{\partial r^*}(r^*\frac{\partial \theta}{\partial x^*}) \tag{1.4}$$

and the boundary conditions are as follows:

$\theta(1, x^*) = 0$ for $x^* > 0$

$\theta(r^*, 0) = 1$ for $x^* \le 0$

The solution for this system can be obtained (Graetz):

$$\theta(r^*, x^*) = \sum_{n=1}^{\infty} C_n G_n(r^*) e^{-\lambda_n{}^2 x^*} \tag{1.5}$$

where the λ_n are the eigenvalues required to make the solution satisfy the following differential equation :

$$r^* G_n'' + G_n' + \lambda_n{}^2 r^* (1-r^{*2}) G_n = 0 \tag{1.6}$$

Graetz posed an eigenfunction:

$$G_n(\lambda_n) = \sum_{k=0}^{\infty} \lambda_n^{2k} d_k = 1 + \lambda_n^2 d_1 + \lambda_n^4 d_2 + \lambda_n^6 d_3 + \ldots \tag{1.7}$$

and he gave only two values: $\lambda_1 = 2.704$ and $\lambda_2 = 6.50$. Unfortunately, it was very difficult at that time to calculate the larger values of λ_n.

Sellars et al.(1956) extended the problem to include a more effective approximation technique for evaluation of the eigenvalues of the problem. They developed an approximate method by using three expressions to represent the Graetz functions in three ranges: (1) near the center; (2) between the centerline and the wall; and (3) near the wall. They obtained an approximate expression as follows:

$$\lambda_n = 4\,(n-1) + 8/3 \qquad n = 1,2,3,\ldots\ldots \tag{1.8}$$

The comparison of the values with other investigations (see Table 5.3) shows that the approximate method is correct and effective, especially for larger n. The accuracy, except the first eigenvalue, is acceptable. This work solved the Graetz problem completely.

The objective of this research is to evaluate the eigenvalues for the Graetz problem in slip–flow — a heat transfer problem for gases at low pressures or in extremely small tubes. To do this, the velocity profile with slip–flow must be found first, and a mathematical model of temperature distribution in slip flow must be established by combining the energy and momentum equations. Next, by using the method of Frobenius, a series solution must be obtained and finally, a technique for evaluation of the eigenvalues for the series solution must be developed. For practical calculations, relationships between the eigenvalues and the Knudsen number should be obtained.

CHAPTER 2

VELOCITY AND TEMPERATURE DISTRIBUTIONS

In order to build the mathematical model for the Graetz Problem in slip–flow, the velocity profile must be found first. In this chapter, based on some assumptions, the expression for velocity will be derived from the continuity equation and momentum equation. The slip condition will be used to evaluate the slip velocity and the velocity will be expressed in terms of Knudsen numbers. A mathematical model of temperature distribution in slip–flow will be established by combining the energy and momentum equations.

2.1 Velocity Distribution

As a model, one can consider the flow of a fluid in a circular tube of radius R, shown in Figure 2.1:

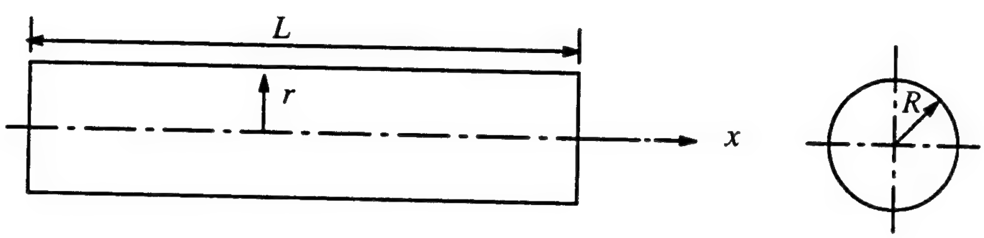

Fig. 2.1 Coordinate system for the problem

For this model the following conditions have been assumed (Barron, 1994):

(1) The flow is steady. This means that the properties of the flow are time independent.

(2) The fluid is incompressible (or, if a gas is considered, the Mach number is low). In this case, the density may be assumed constant.

(3) The flow is fully established. In this case, the axial velocity, u, is a function of the radial coordinate only, and not a function of the axial coordinate. In addition, the radial velocity is zero.

(4) The "swirl" component of velocity is identically zero. This fact means that the flow properties are independent of the angular coordinate in cylindrical coordinates.

(5) Fluid properties are constant.

(6) Energy dissipation effects are negligible.

(7) The tube wall temperature is constant.

2.1.1 Continuity Equation

The general continuity equation can be written in cylindrical coordinates as follows:

$$\frac{\partial \varrho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\varrho r v) + \frac{\partial}{\partial x}(\varrho u) = 0 \tag{2.1}$$

For steady flow of an incompressible fluid, Eq. (2.1) reduces to:

$$\frac{1}{r}\frac{\partial}{\partial r}(rv) + \frac{\partial u}{\partial x} = 0$$

For fully developed flow,

$$\frac{\partial u}{\partial x} = 0$$

Therefore,

$$r\,v = \text{constant}$$

Since the radial velocity is zero at the wall (the wall is impermeable), we must conclude that:

$$v = 0 \text{ (identically)}.$$

2.1.2 Velocity Distribution with Slip Condition

The Momentum Equation can be written in cylindrical coordinates as follows (Kays et al., 1993):

$$\varrho u \frac{\partial u}{\partial x} + \varrho v \frac{\partial u}{\partial r} + \frac{dp}{dx} = \frac{\mu}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) \tag{2.2}$$

For fully–established, steady–flow of an incompressible fluid, the Navier–Stokes equation for the axial direction reduces to:

$$-\frac{dp}{dx} + \frac{\mu}{r} \frac{d}{dr}\left(r \frac{du}{dr}\right) = 0 \tag{2.3}$$

In this case, the velocity u is independent of x, so we can define the constant,

$$C_1 = -\frac{1}{4\mu} \frac{dp}{dx} \tag{2.4}$$

Then, Eq. (2.3) can be written in the following form:

$$4C_1 + \frac{1}{r} \frac{d}{dr}\left(r \frac{du}{dr}\right) = 0 \tag{2.5}$$

This expression can be solved by directly integrating twice to yield:

$$u = C_2 \ln r + C_3 - C_1 r^2 \tag{2.6}$$

Since the velocity u is finite at the center of the tube ($r=0$), we must have $C_2 = 0$.

At the centerline of the tube ($r=0$), the velocity is equal to u_c (the center–line velocity). Using this condition in Eq. (2.6), we obtain: $u(0) = u_c = C_3$

At the surface of the tube ($r=R$), the velocity is not zero in slip–flow, but is equal to a finite velocity u_s. Using this condition, we find, from Eq. (2.6):

$$u(R) = u_s = u_c - C_1 R^2 \tag{2.7}$$

Rearranging and solving for C_1:

$$C_1 = \frac{u_c - u_s}{R^2} = -\frac{1}{4\mu}\frac{dp}{dx} \tag{2.8}$$

Making these substitutions, we find the following expression for the velocity distribution:

$$u = u_c - (u_c - u_s)(r/R)^2 = u_c\,[\,1 - (r/R)^2\,] + u_s\,(r/R)^2 \tag{2.9}$$

This velocity profile is shown in Figure 2.2.

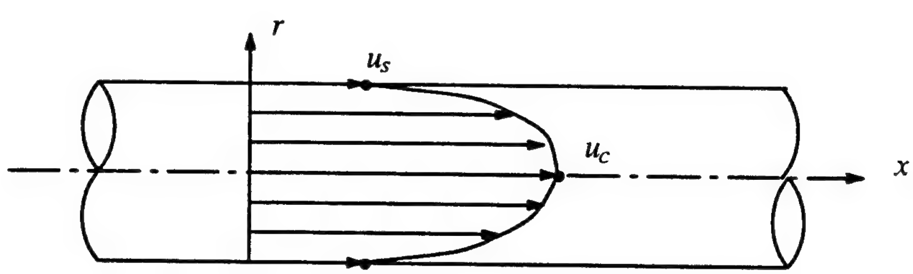

Fig. 2.2 Velocity distribution

Let us now calculate the mean fluid velocity u_m in the tube. The volumetric flow rate can be written as follows:

$$\pi R^2 u_m = \int_0^R 2\pi r u dr \tag{2.10}$$

If we introduce the dimensionless variable, r^*, we can write Eq. (2.10) as:

$$u_m = 2\int_0^1 u r^* dr^* = 2\int_0^1 [u_c - (u_c - u_s) r^{*2}] r^* dr^* \tag{2.11}$$

Carrying out the integration, we obtain:

$$u_m = 2[\ (u_c/2)\ r^{*2} - (u_c - u_s)\ (r^{*4}/4)\ ]\ \Big|_0^1$$

or

$$u_m = 2[\, u_c/2 - (\, u_c - u_s)/4\,] = (\, u_c + u_s)/2 \tag{2.12}$$

The centerline velocity can be written in terms of the mean velocity and the slip velocity, as follows:

$$u_c = 2\ u_m - u_s \tag{2.13}$$

Making this substitution, we find the following expression for the velocity profile in slip–flow:

$$u = 2(\, u_m - u_s)\,(1 - r^{*2}) + u_s \tag{2.14}$$

If the slip velocity is zero, then Eq. (2.14) reduces to the Poiseuille distribution:

$$u = 2\, u_m\ (1 - r^{*2}) \tag{2.15}$$

2.1.3 Evaluation of the Slip Velocity

The slip velocity can be evaluated from Eq. (1.1) as:

$$u_s = -\frac{\lambda}{R}\left(\frac{du}{dr*}\right)_{r*=1} = +\frac{4\lambda(u_m-u_s)}{R} = \frac{8\lambda(u_m-u_s)}{D} \tag{2.16}$$

Introducing the Knudsen number, we find:

$$\frac{u_s}{u_m} = \frac{8Kn}{1 + 8Kn} \tag{2.17}$$

Making this substitution in Eq. (2.14), we obtain the following expression for the velocity profile in slip flow:

$$\frac{u}{u_m} = \frac{2(1-r^{*2}) + 8Kn}{1 + 8Kn} \tag{2.18}$$

The molecular mean free path for a gas can be calculated from the following expression(Sreekanth, 1968):

$$\lambda = \frac{\mu}{\varrho}\left(\frac{\pi}{2R_gT}\right)^{1/2} \tag{2.19}$$

The acoustic velocity for a gas is given by:

$$c_a = (\gamma R_g T)^{1/2} \tag{2.20}$$

where: $\gamma = c_p/c_v$. Substituting Eq. (2.20) into Eq.(2.19), we obtain:

$$\lambda = \frac{\mu}{\varrho c_a}\left(\frac{\pi\gamma}{2}\right)^{1/2} = \frac{1.4829\mu}{\varrho c_a} \tag{2.21}$$

for a gas having a specific heat ratio, $\gamma = 1.40$. The Knudsen number may then be written as the reciprocal of the Reynolds number based on the sonic velocity in the gas as follows:

$$Kn = \frac{\mu}{D\varrho c_a}\left(\frac{\pi\gamma}{2}\right)^{1/2} = \frac{1}{Re*}\left(\frac{\pi\gamma}{2}\right)^{1/2} = \frac{1}{0.6743\ Re*} \tag{2.22}$$

From the expression, it is obvious that the Knudsen number is dependent on $Re*$ for a gas. Thus, for a given temperature, Kn can be calculated from Eq. (2.22) and the velocity distribution can be determined from Eq. (2.18).

For example, for nitrogen gas at 300 K (26.8°C or 80°F) and atmospheric pressure, the gas mean free path may be determined from Eq. (2.21), where the property values for nitrogen gas are as follows:

$\varrho = 1.6332$ kg/m^3 = 0.07106 lbm/ft^3

μ =0.01784 mPa–s = 0.04316 lbm /ft–hr

R_g = 206.8 J/kg–K = 55.15 ft–lbf/lbm–°R

$$\lambda = \frac{(0.01784)\ (10^{-3})}{(1.6332)}\left[\frac{\pi}{(2)\ (206.8)\ (300)}\right]^{1/2}$$

$\lambda = 0.2189 \times 10^{-6}$ m = 0.2189 μm

If we accept a difference of 5 percent between the case for slip flow and the case for continuum flow as the effect of the slip condition, from Eq. (2.17) we find that the slip flow effects become significant when:

$$\frac{u_s}{u_m} = \frac{8Kn}{1 + 8Kn} = 0.05$$

Then

$$8Kn \approx 0.05 \text{ , or } Kn \approx 0.00625$$

The corresponding tube diameter is:

$$D = 0.2189/0.00625 = 35.0\ \mu m$$

For slip–flow ($10^{-3} < Kn < 0.1$), the tube diameter range is:

$$D_{min} = 2.2\ \mu m \quad \text{for } Kn=0.1$$

$$D_{max} = 218.9\ \mu m \quad \text{for } Kn=0.001$$

and from Eq. (2.18) the maximum velocity can be found in the range of:

$$u_{max} / u_m = 1.556 \quad \text{for } Kn = 0.1$$

$$u_{max} / u_m = 1.992 \quad \text{for } Kn = 0.001$$

while $\quad u_{max} / u_m = 2 \quad$ for $Kn = 0$ (no slip)

which means that the velocity difference between the wall and the centerline can be reduced significantly in small size tubes.

2.2 Temperature Distribution

The general temperature field equation for flow of an incompressible fluid with zero swirl or angular components, zero energy generation, and negligible frictional energy dissipation is as follows:

$$\varrho c \left(\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial r} + u\frac{\partial T}{\partial x}\right) = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial x^2}\right] \tag{2.23}$$

For steady flow and for zero radial velocity ($v = 0$), the energy equation reduces to:

$$u\frac{\partial T}{\partial x} = \alpha \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r} \right) + \frac{\partial^2 T}{\partial x^2} \right] \tag{2.24}$$

From an order–of–magnitude analysis for the case in which the tube length is much larger than the tube diameter, the second term on the right side of Eq. (2.24) is much smaller than the first term on the right side and the energy equation can then be written in the following form:

$$u\frac{\partial T}{\partial x} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r} \right) \tag{2.25}$$

One can then consider the case for flow which is fully established hydrodynamically at the end of the insulated section (a long entry length), but which is developing thermally due to the temperature jump on the tube wall. The physical model is illustrated in Figure 1.1. The velocity profile is fully–established at the end of the insulated section, $x=0$, and the temperature of the fluid entering the uninsulated section is uniform $T = T_o$. The boundary conditions for this situation are:

$$T(R, x) = T_w \quad \text{for } x > 0$$

$$T(r, 0) = T_0 \quad \text{for } x \leq 0$$

Using the dimensionless variables, the energy equation may be written in the following form:

$$\frac{R^2 u}{\alpha L}\frac{\partial \theta}{\partial x*} = \frac{1}{r*}\frac{\partial}{\partial r*}\left(r*\frac{\partial \theta}{\partial r*} \right) \tag{2.26}$$

Using the velocity distribution given by Eq. (2.14)

$$u = 2(\, u_m - u_s)\,(1 - r*^2) + u_s = 2(\, u_m - u_s)\,(1 - r*^2 + 4Kn) \tag{2.27}$$

the energy equation reduces to:

$$\frac{u_m D^2}{2\alpha L(\, 1 + 8Kn)}\frac{\partial \theta}{\partial x*} = \frac{1}{(1 - r*^2 + 4Kn) r*}\frac{\partial}{\partial r*}\left(r*\frac{\partial \theta}{\partial r*} \right) \tag{2.28}$$

The Graetz number Gz is defined by:

$$Gz = Re\ Pr\ (D/L) = \frac{D u_m \varrho}{\mu}\frac{\mu c}{k}\frac{D}{L} = \frac{u_m D^2}{\alpha L}$$

The energy equation, Eq. (2.28), can be written as follows:

$$\frac{Gz}{2(\, 1 + 8Kn)}\frac{\partial \theta}{\partial r*} = \frac{1}{(1 - r*^2 + 4Kn) r*}\frac{\partial}{\partial r*}\left(r*\frac{\partial \theta}{\partial r*} \right) \tag{2.29}$$

with boundary conditions:

$$\theta(1, x*) = 0 \qquad \text{for } x* > 0$$

$$\theta(r*, 0) = 1 \qquad \text{for } x* \le 0$$

2.3 Summary

In this chapter, the velocity distribution with slip–flow has been obtained. It can be expressed simply in terms of the Knudsen number. From the expression, it is obvious that the velocity increases always as Kn increases. From the relationship of Kn in term of molecular mean free path for a gas λ and diameter of tube D, we can see that Kn in microtubes

may become large enough to significantly affect the velocity distribution and consequently affect the heat transfer for this problem. Also, a mathematical model of temperature distribution in slip–flow has been established by combining the energy and momentum equations.

CHAPTER 3

ANALYTICAL SOLUTION

In the last chapter, the velocity distribution was expressed in terms of mean velocity and Knudsen number, and a mathematical model of temperature distribution in slip–flow was established by combining the energy and momentum equations. In this chapter, a series solution will be obtained by the method of Frobenius. Considering the given boundary condition, a temperature distribution in terms of a generalized Fourier series will be derived. Also, expressions for the local and overall Nusselt numbers will be obtained.

3.1 Graetz Solution

3.1.1 Separation of Variables Solution

Eq. (2.29) can be solved by a separation–of–variables technique. Suppose we let

$$\theta(r^*,x^*) = G(r^*)\, X(x^*)$$

Making this substitution into Eq. (2.29) and rearranging the components results in the following:

$$\frac{Gz}{2(\ 1 + 8Kn)\ X}\ \frac{dX}{dx^*} = \frac{1}{(1-r^{*2} + 4Kn)r^*}\frac{d}{dr^*}\left(r^*\frac{dG}{dr^*} \right) = -\lambda^2 \qquad (3.1)$$

where λ is an arbitrary constant. The ordinary differential equations which result are:

$$\frac{dX}{dx^*} + \frac{2(\ 1 + 8Kn)\lambda^2}{Gz}X = 0 \qquad (3.2)$$

and

$$\frac{d^2G}{dr^{*2}} + \left(\frac{1}{r^*}\frac{dG}{dr^*} \right) + \lambda^2(1-r^{*2} + 4Kn)G = 0 \tag{3.3}$$

with boundary conditions:

$G(1) = 0,$

and

$G(0) = 1.$

The solution of Eq. (3.2) is:

$$X(x^*) = C\, exp\left[-\frac{2(1 + 8Kn)\lambda^2}{Gz} x^* \right] \tag{3.4}$$

The constant C in Eq. (3.4) will be evaluated below, and λ will be evaluated in the next chapter.

The solution of Eq. (3.3) may be obtained by the method of Frobenius. Suppose we take the function $G(r^*)$ as a power series.

$$G(r^*) = \sum_{j=0}^{\infty} a_j\, r^{*j} \tag{3.5}$$

Then,

$$G'(r^*) = \sum_{j=1}^{\infty} j a_j\, r^{*j-1} = \sum_{j=0}^{\infty} (j + 1)a_{j+1}\, r^{*j} \tag{3.6}$$

and

$$G''(r^*) = \sum_{j=1}^{\infty} (j + 1)j\, a_{j+1}\, r^{*j-1} = \sum_{j=0}^{\infty} (j + 1)(j + 2)a_{j+2}\, r^{*j} \tag{3.7}$$

Making these substitutions into Eq. (3.3), we obtain:

$$\sum_{j=0}^{\infty} (j + 1)(j + 2)a_{j+2}\, r^{*j} + \sum_{j=0}^{\infty} (j + 1)\, a_{j+1}\, r^{*j-1} +$$

$$+ \lambda^2\left[(1 + 4Kn) \sum_{j=0}^{\infty} a_j\, r^{*j} - \sum_{j=0}^{\infty} a_j\, r^{*j+2} \right] = 0 \tag{3.8}$$

Expanding Eq. (3.8) and multiplying each term by r^*, we obtain:

$$(2a_2 r^* + 6a_3 r^{*2} + 12a_4 r^{*3} + \ldots) + (a_1 + 2a_2 r^* + 3a_3 r^{*2} + 4a_4 r^{*3} + \ldots) +$$

$$+ \lambda^2[\ (1 + 4Kn\ )\ (a_0 r^* + a_1 r^{*2} + a_2 r^{*3} + \ldots) - \ (a_0 r^{*3} + \ \ldots)] \ = \ 0 \qquad (3.9)$$

The first two constants a_0 and a_1 are arbitrary, so let us take the even solution with the following values:

$$a_0 = 1 \quad \text{and} \quad a_1 = 0$$

Equating the coefficients on like powers of r^* in Eq. (3.9), we obtain:

$$2\,a_2 + 2\,a_2 + \lambda^2 (\,1 + 4Kn\,)\,a_0 = 0$$

or,

$$a_2 = -\,(\,\lambda\,/\,2\,)^2\,(1 + 4\,Kn\,)$$

and

$$6\,a_3 + 3\,a_3 + \lambda^2 (\,1 + 4Kn\,)\,a_1 = 0$$

or,

$$a_3 = 0$$

In fact, we find that all terms involving odd numbered subscripts drop out.

$$a_{2k-1} = 0 \quad \text{for } k = 1,2,3, \ldots$$

For the coefficients a_j, with $j \geq 4$, we find the following recursion relationship:

$$(j-1)(j)\,a_j + j\,a_j + \lambda^2[\,(1+4Kn)\,a_{j-2} - a_{j-4}\,] = 0$$

or,

$$a_j = -(\lambda/j)^2[\,(1+4Kn)\,a_{j-2} - a_{j-4}\,] \quad \text{for } j = 4, 6, 8, \ldots \tag{3.10}$$

3.1.2 Eigenfunction of the Series Solution

At the surface of the tube ($r^* = 1$), the temperature in slip flow is given as follows (Eckert and Drake, 1972):

$$T_s - T_w = -\left(\frac{2\gamma}{1+\gamma}\right)\frac{\lambda}{Pr}\left(\frac{\partial T}{\partial r}\right)_{r=R} \tag{3.11}$$

Introducing the dimensionless variables, this condition can be written as follows:

$$\theta_s = \left(\frac{T_s - T_w}{T_0 - T_w}\right) = -\frac{4\,\gamma}{1+\gamma}\,\frac{Kn}{Pr}\left(\frac{\partial\theta}{\partial r^*}\right)_{r^*=1} \tag{3.12}$$

We note that:

$$\theta_s = \theta\,(1,x^*) = X(x^*)\,G(1)$$

and

$$\frac{\partial\theta(1,x^*)}{\partial r^*} = X(x^*)\,\frac{dG(1)}{dr^*}$$

Therefore,

$$G(1) = -\frac{4\,\gamma}{1+\gamma}\,\frac{Kn}{Pr}\,\frac{dG(1)}{dr^*} \tag{3.13}$$

We can write Eq. (3.5) and Eq. (3.6) as follows:

$$G\,(r^*) = \sum_{j=0}^{\infty} a_{2j}\, r^{*2j} = 1 + a_2\, r^{*2} + a_4\, r^{*4} + \ldots$$

$$\frac{dG(r^*)}{dr^*} = \sum_{j=1}^{\infty} 2j\, a_{2j}\, r^{*2j-1} = 2\,a_2\, r^* + 4\,a_4\, r^{*3} + \ldots$$

Making these substitutions into Eq. (3.13), where $dG(1)/dr^* = dG(r^*)/dr^*|_{r^*=1}$, we obtain the following condition:

$$1 + \sum_{j=1}^{\infty} a_{2j} \left[1 + 2j \left(\frac{4\gamma}{1+\gamma} \right) \frac{Kn}{Pr} \right] = 0 \qquad (3.14)$$

Eq. (3.14) defines the present problem. The coefficients a_{2j} are functions of the eigenvalues λ_n, where n =1,2,3, ... The eigenfunctions for this problem can be written as follows:

$$G_n(r^*) = \sum_{j=0}^{\infty} a_{2j} \; (\lambda_n) \; r^{*2j} \qquad n = 1, 2, 3, ... \qquad (3.15)$$

3.1.3 Determination of Constants C_n

We can write the solution for the temperature distribution in terms of a generalized Fourier series, as follows:

$$\theta \; (r^*, x^*) = \sum_{n=1}^{\infty} C_n G_n(r^*) \; exp \left[- \frac{2 \; (\lambda_n)^2 \; x^* \; (1 + 8 \; Kn)}{Gz} \right] \qquad (3.16)$$

Note that the lower limit on n is now 1, which is arbitrary.

The constants C_n can be found from the entrance condition,

at $x = 0$ (or $x^* = 0$); $T(r, 0) = T_0$, or $\theta (r^*, 0) = 1$

Making this substitution into Eq. (3.16), we obtain:

$$\sum_{j=0}^{\infty} C_n \;\; G_n(r^*) = 1 \qquad (3.17)$$

The governing differential equation, along with the boundary conditions, is a Sturm–Liouville problem, with a weight function,

$$w = r^*(1 - r^{*2} + 4\,Kn)$$

and from orthogonality,

$$\int_0^1 G_n G_m r^* (1-r^{*2} + 4Kn)dr^* = 0 \quad for \;\; m \neq n$$

The constants may be evaluated by multiplying Eq. (3.17) on both sides by

$$G_m r^* (1 - r^{*2} + 4Kn)$$

and integrating between 0 and 1. Only the term in which m = n is non–zero, and we find:

$$\int_0^1 G_n \; r^* (1-r^{*2} + 4Kn) \; dr^* = C_n \int_0^1 (G_n)^2 r^* (1-r^{*2} + 4Kn) \; dr^* \quad (3.18)$$

The integral on the left side may be evaluated from the differential equation (Eq. (3.3)), as follows:

$$G_n \; r^* (1-r^{*2} + 4Kn) = - \frac{1}{(\lambda_n)^2} \frac{\partial}{\partial r^*} (r^* \frac{\partial G_n}{\partial r^*})$$

and

$$\int_0^1 G_n \; r^* (1-r^{*2} + 4Kn) \; dr^* = - \frac{1}{(\lambda_n)^2} (r^* \frac{\partial G_n}{\partial r^*}) \Big|_0^1$$

which becomes

$$\int_0^1 G_n \; r^* (1-r^{*2} + 4Kn) \; dr^* = - \frac{1}{(\lambda_n)^2} (\frac{\partial G_n}{\partial r^*})_{r^* = 1} \quad (3.19)$$

The integral on the right side of Eq. (3.18) can be evaluated from (Graetz, 1885, see Appendix D)

$$\int_0^1 (G_n)^2 r^* \left(1 - r^{*2} + 4Kn \right) dr^* = \frac{1}{2\lambda_n}\left[\left(\frac{\partial G}{\partial \lambda} \right)_n \frac{\partial G_n}{\partial r^*} \right]_{r^*=1}$$

Making the substitutions for the integrals, we find the following values for the constants in the series expansion:

$$C_n = -\frac{2}{\lambda_n \left[\left(\frac{\partial G}{\partial \lambda} \right)_n \right]_{r^*=1}} \tag{3.20}$$

The term in the denominator may be evaluated, as follows:

$$\left[\left(\frac{\partial G}{\partial \lambda} \right)_n \right]_{r^*=1} = \left[\sum_{j=0}^{\infty} \left(\frac{da_{2j}}{d\lambda} \right)_n r^{*2j} \right]_{r^*=1} = \sum_{j=0}^{\infty} \left(\frac{da_{2j}}{d\lambda} \right)_n$$

Each of the terms in this equation may be worked out from the previous results, Eq. (3.10), as follows:

$$\left(\frac{da_2}{d\lambda} \right)_n = -\left(\frac{\lambda_n}{2} \right) \left(1 + 4\,Kn \right)$$

$$\left(\frac{da_4}{d\lambda} \right)_n = \left(\frac{\lambda_n}{8} \right) \left[1 + \frac{\lambda_n^2}{2} \left(1 + 4\,Kn \right)^2 \right]$$

$$\left(\frac{da_6}{d\lambda} \right)_n = -\frac{\lambda_n^3 \left(1 + 4\,Kn \right)}{144} \left[5 + \frac{3\lambda_n^2}{8} \left(1 + 4\,Kn \right)^2 \right]$$

....

For the special case of x/L/Gz > 0.05, which means that the entrance effect can be neglected for the rest of the tube, only the first term in Eq. (3.15) is needed to represent the temperature distribution accurately. In this case, we have:

$$\theta\ (r^*,x^*) = \frac{T(r,x) - T_w}{T_0 - T_w} = C_1 G_1(r^*)\ exp[- \frac{2\ (\lambda_1{}^2)\ x^*\ (1 + 8\ Kn)}{Gz}] \qquad (3.21)$$

The expressions for the temperature distribution are summed as follows:

$$\theta\ (\ r^*,x^*) = \sum_{n=1}^{\infty} C_n G_n(r^*)\ exp\ [\ - \frac{2\ (\lambda_n)^2\ x^*\ (1 + 8\ Kn)}{Gz}\ ] \qquad (3.16)$$

where

$$G_n(r^*)\ = \sum_{j=0}^{\infty} a_{2j}\ (\lambda_n)\ r^{*2j} \qquad n\ = 1, 2, 3,\ ... \qquad (3.15)$$

$$C_n = - \frac{2}{\lambda_n [\ (\ \frac{\partial G}{\partial \lambda}\ )_n\ ]_{r^* = 1}} \qquad (3.20)$$

$$[\ (\ \frac{\partial G}{\partial \lambda}\ )_n\ ]_{r^* = 1}\ = \sum_{j=0}^{\infty} (\frac{da_{2j}}{d\lambda})_n$$

From these expressions, we can see that the coefficients a_{2j} and λ_n must be predetermined in order to calculate the temperature.

3.2 Heat Transfer Coefficient Correlation

3.2.1 Bulk Temperature

The bulk or average temperature can be determined from:

$$\theta_B = \frac{2}{R^2}\int_0^R (u/u_m)\ \theta\ r\ dr$$

or

$$\theta_B = 2\cdot\int_0^1 (u/u_m)\ \theta\ r^*\ dr^* \tag{3.22}$$

where:

$$\theta_B = \frac{T_B - T_w}{T_0 - T_w}$$

Using the velocity distribution from Eq. (2.18) and the temperature distribution from Eq. (3.16), we can evaluate the bulk temperature at any location along the length of the tube, as follows:

$$\theta_B = 4\sum_{n=1}^{\infty}\frac{C_n}{(1+8Kn)}\ exp\ [-\frac{2\ (\lambda_n)^2\ x^*\ (1+8\ Kn)}{Gz}]\cdot$$

$$\int_0^1 G_n\ (\ 1 - r^{*2} + 4Kn\ )\ r^*\,dr^* \tag{3.23}$$

The integral had been worked out previously in Eq. (3.19). Making this substitution, we obtain the expression for the local bulk temperature of the fluid in the tube.

$$\theta_B = -\frac{4}{(1+8Kn)}\sum_{n=1}^{\infty}\frac{C_n}{(\lambda_n)^2}\frac{dG_n(1)}{dr^*}exp\ [-\frac{2\ (\lambda_n)^2\ x^*\ (1+8\ Kn)}{Gz}\ ] \tag{3.24}$$

3.2.2 Local Heat Transfer Coefficient

The local or "point" convective heat transfer coefficient can be defined by:

$$Q/A_w = h_x \, (T_B - T_w) \tag{3.25}$$

The heat flux can also be written, as follows:

$$\frac{Q}{A_w} = -k \left(\frac{\partial T}{\partial r} \right) \Big|_{r=R} = -\frac{k \, (T_0 - T_w)}{R} \left(\frac{\partial \theta}{\partial r^*} \right) \Big|_{r^*=1} \tag{3.26}$$

Equating the heat flux from Eqs. (3.25) and (3.26), we obtain the expression for the local convective heat transfer coefficient.

$$h_x = -\frac{k}{R \, \theta_B} \left(\frac{\partial \theta}{\partial r^*} \right) \Big|_{r^*=1} = -\frac{2 \, k}{D \, \theta_B} \left(\frac{\partial \theta}{\partial r^*} \right) \Big|_{r^*=1} \tag{3.27}$$

Making the substitutions from Eq. (3.16) for the temperature gradient at the wall and Eq. (3.24) for the local bulk temperature, the following expression is obtained for the local or "point" Nusselt number.

$$Nu_x = \frac{h_x \, D}{k} = \frac{- \sum_{n=1}^{\infty} 2C_n \frac{dG_n(1)}{dr^*} exp \left[- \frac{2 \, (\lambda_n)^2 \, x^* \, (1+8 \, Kn)}{Gz} \right]}{\theta_B} \tag{3.28}$$

Or,

$$Nu_x = \frac{(1 + 8Kn) \sum_{n=1}^{\infty} C_n \frac{dG_n(1)}{dr^*} exp \left[- \frac{2 \, (\lambda_n)^2 \, x^* \, (1+8 \, Kn)}{Gz} \right]}{2 \sum_{n=1}^{\infty} \frac{C_n}{(\lambda_n)^2} \frac{dG_n(1)}{dr^*} exp \left[- \frac{2 \, (\lambda_n)^2 \, x^* \, (1+8 \, Kn)}{Gz} \right]} \tag{3.29}$$

For the special case of x*/Gz > 0.05, Eq. (3.29) reduces to the following expression:

$$Nu_x = (1/2)(1 + 8\,Kn)(\lambda_1)^2 \tag{3.30}$$

3.2.3 Overall Convective Heat Transfer Coefficient

The average or overall convective heat transfer coefficient is defined through the following expression:

$$Q = \bar{h}_c\,(\pi\,D\,L)\,(\Delta\,T)_{LN} = \int_0^L h_x\,(T_0 - T_w)\,(\pi\,D\,dx) \tag{3.31}$$

where $(\Delta T)_{LN}$ = log–mean–temperature difference (LMTD).

The LMTD may be written in terms of the inlet temperature T_0 and the exit bulk temperature T_L, as follows:

$$(\Delta\,T)_{LN} = \frac{(T_0 - T_w) - (T_L - T_w)}{ln\,[\,(T_0 - T_w)\,/\,(T_L - T_w)\,]} = \frac{(\theta_{B,L} - 1)\,(T_0 - T_w)}{ln\,(\theta_{B,L})} \tag{3.32}$$

Let us define the dimensionless LMTD, as follows:

$$\theta_{LN} = \frac{(\Delta\,T)_{LN}}{T_0 - T_w} \tag{3.33}$$

Then,

$$\theta_{LN} = \frac{(\theta_{B,L} - 1)}{ln\,(\theta_{B,L})} \tag{3.34}$$

The expression for the average convective heat transfer coefficient can then be written, from Eq. (3.31):

$$\bar{h}_c = \frac{1}{\theta_{LN}} \int_0^1 h_x\,\theta_B\,dx^* \tag{3.35}$$

We had obtained the following result previously in Eq. (3.28):

$$h_x\theta_B = -\frac{2\ k}{D}\ (\ \frac{\partial\theta}{\partial r*}\ )\ |_{\ r*\ =\ 1}$$

and

$$h_x\ \theta_B = -\sum_{n=1}^{\infty} \frac{2k\ C_n}{D}\ \frac{dG_n(1)}{dr*} exp\ [\ -\ \frac{2\ (\lambda_n)^2\ x*(1\ +\ 8\ Kn)}{Gz}\ ] \tag{3.36}$$

Making this substitution into Eq. (3.35) and integrating, we obtain the following result for the average Nusselt number:

$$\overline{Nu} = \frac{\bar{h}_c\ D}{k}$$

$$\overline{Nu} = -\frac{Gz}{\theta_{LN}(1\ +\ 8Kn)} \sum_{n=1}^{\infty} \frac{C_n}{(\lambda_n)^2}\ \frac{dG_n(1)}{dr*}\{1-exp[-\frac{2\ (\lambda_n)^2\ x*\ (1\ +\ 8\ Kn)}{Gz}]\} \tag{3.37}$$

The expression for the average Nusselt number may be written in a somewhat more compact form, as follows. At the inlet of the tube, the temperature of the fluid is uniform,

$$T\,(\ r,\ 0\ ) = T_0\ ,\quad \theta_{B,0} = 1.$$

Using Eq. (3.24), we obtain:

$$\theta_{B,0} = 1 = \frac{4}{(1\ +\ 8Kn)} \sum_{n=1}^{\infty} \frac{C_n}{(\lambda_n)^2}\ \frac{dG_n(1)}{dr*} \tag{3.38}$$

and,

$$\theta_{B,L} = -\frac{4}{(1 + 8Kn)} \sum_{n=1}^{\infty} \frac{C_n}{(\lambda_n)^2} \frac{dG_n(1)}{dr^*} \; exp[-\frac{2 \; (\lambda_n)^2 \; x^* \; (1 + 8 \; Kn)}{Gz}] \tag{3.39}$$

By comparison with the expression given in Eq. (3.37) and using Eq. (3.34), we obtain:

$$\overline{Nu} = \frac{Gz \; (\; 1 - \theta_{B,L} \;)}{4 \; \theta_{LN}} = -\frac{1}{4} \; Gz \; ln \; (\; \theta_{B,L} \;) \tag{3.40}$$

The expressions for both local and average Nusselt number are summed as follows:

$$Nu_x = \frac{(1 + 8Kn) \sum_{n=1}^{\infty} C_n \frac{dG_n(1)}{dr^*} exp \; [- \frac{2 \; (\lambda_n)^2 \; x^* \; (1+8 \; Kn)}{Gz} \;]}{2 \sum_{n=1}^{\infty} \frac{C_n}{(\lambda_n)^2} \frac{dG_n(1)}{dr^*} exp \; [- \frac{2 \; (\lambda_n)^2 \; x^* \; (1+8 \; Kn)}{Gz} \;]} \tag{3.29}$$

$$\overline{Nu} = -\frac{Gz}{\theta_{LN}(1 + 8Kn)} \sum_{n=1}^{\infty} \frac{C_n}{(\lambda_n)^2} \frac{dG_n(1)}{dr^*} \{ 1 - exp[- \frac{2 \; (\lambda_n)^2 \; x^* \; (1 + 8 \; Kn)}{Gz}] \} \tag{3.37}$$

$$\theta_{LN} = \frac{(\; \theta_{B,L} - 1 \;)}{ln \; (\; \theta_{B,L} \;)} \tag{3.34}$$

$$\theta_{B,L} = -\frac{4}{(1 + 8Kn)} \sum_{n=1}^{\infty} \frac{C_n}{(\lambda_n)^2} \frac{dG_n(1)}{dr^*} \; exp[-\frac{2 \; (\lambda_n)^2 \; x^* \; (1 + 8 \; Kn)}{Gz}] \tag{3.39}$$

$$G_n(r^*) = \sum_{j=0}^{\infty} a_{2j} \; (\lambda_n) \; r^{*2j} \qquad n = 1, 2, 3, \; ... \tag{3.15}$$

$$C_n = -\frac{2}{\lambda_n\left[\left(\frac{\partial G}{\partial \lambda}\right)_n\right]_{r^*=1}} \tag{3.20}$$

$$\left[\left(\frac{\partial G}{\partial \lambda}\right)_n\right]_{r^*=1} = \sum_{j=0}^{\infty}\left(\frac{da_{2j}}{d\lambda}\right)_n$$

$$\frac{dG_n(1)}{dr^*} = \frac{dG_n(r^*)}{dr^*}\Big|_{r^*=1} = \sum_{j=1}^{\infty} 2j\, a_{2j}$$

From these expressions we can see that the coefficient a_{2j} and λ_n must be predetermined in order to calculate the Nusselt numbers.

3.3 Summary

In this chapter, a series solution for the mathematical model of temperature distribution in slip flow in circular geometries has been obtained by the method of Frobenius. Considering the given boundary condition, a temperature distribution in terms of a generalized Fourier series has been derived. Also, expressions for the local and overall Nusselt numbers have been obtained. All these expressions can be taken as functions of the Knudsen number and the Graetz number. In order to calculate either the temperature or the Nusselt numbers, the coefficient a_{2j} and λ_n must be predetermined.

CHAPTER 4

EVALUATION OF EIGENVALUES

In the last chapter, we obtained a series solution for the temperature distribution. Also, expressions for the local and overall Nusselt numbers have been obtained, as functions of the Knudsen number and the Graetz number. In this chapter we presents a technique for expansion of the coefficient a_{2j} and evaluation of eigenvalues for the solution of the Graetz Problem in slip–flow, since the coefficient a_{2j} and eigenvalues λ_n must be predetermined for the calculation of either the temperature distribution or the Nusselt numbers. A matrix will be constructed and a formulation described to find the coefficients a_{2j} directly as well as d_k. Based on these d_k, the eigenvalues λ_n can be calculated easily.

4.1. Introduction

The series solution of Eq. (3.8) can be expressed as Eq. (3.16), which required the solution of Eq. (3.10).

Letting $\beta = (1+4Kn)$, Eq. (3.10) can be rewritten as follows:

$$\frac{d^2G}{dr^{*2}} + \left(\frac{1}{r^*}\frac{dG}{dr^*} \right) + \lambda^2(\beta - r^{*2})G = 0 \tag{4.1}$$

with boundary conditions:

$$G(0) = 1 \tag{4.2}$$

$$G(1) = 0 \tag{4.3}$$

A particular solution of Equation (4.1) satisfying condition (4.2) is

$$G(r^*) = \sum_{k=0}^{\infty} a_k r^{*2k} \tag{4.4}$$

Substituting $j = 2k$, the recursion relationship Eq.(3.10) can be rewritten as follows:

$$\left.\begin{aligned} a_0 &= 1\,, \quad 4a_1 + \beta\lambda^2 = 0 \\ 4k^2 a_k &+ \lambda^2[\beta a_{k-1} - a_{k-2}] = 0 \end{aligned}\right\} \qquad (4.5)$$

for $k = 2,3,4,\ldots$.

From Eq. (4.5) we can see that the expansion of a_k involves λ with different powers. In order to satisfy Eq. (4.3), the parameter λ must take on an infinite number of discrete values λ_n, and $G(1)$ can be taken as a function of λ. Then, an eigenfunction is obtained by rearranging terms according to λ^{2k}:

$$G(r^*)\,|_{r^*=1} = G(1,\lambda) = \sum_{k=0}^{\infty} d_k \lambda^{2k} = 1 + d_1\lambda^2 + d_2\lambda^4 + d_3\lambda^6 + \ldots = 0 \quad (4.6)$$

One of the procedures employed in determining the values of λ is to (1) expand each coefficient a_k to a large k by relationship Eq. (4.5); (2) add up all the terms with the same order of λ in determining the coefficients d_k in Eq. (4.6); and, (3) determine the values of λ_n. This technique is computationally inefficient for large k because the expansion of a_k for large k is very complicated and makes the determination of λ_n from the eigenfunction $G(1,\lambda) = 0$ extremely difficult. It is the purpose of this chapter to obtain a formulation for calculation of a_k in Eq. (4.5) as well as for λ_n in the eigenfunction expansion (Eq. (4.6)) in a more effective way.

4.2. Formulation of d_k

4.2.1 Expansion of a_k ($Kn = 0$ or $\beta = 1$)

First of all, let us expand the coefficients a_k. For simplification, let coefficients a_k be expanded from the recursion relationship Eq. (4.5) for $\beta = 1$ (or $Kn = 0$) as follows:

$$a_0 = \quad 1$$

$$a_1 = -\,\lambda^2/[2^{(2)(1)}(1!)^2]$$

$$a_2 = \quad [\,\lambda^2 2^2+\lambda^4]/[2^{(2)(2)}(2!)^2]$$

$$a_3 = -[\,\lambda^4(2^2+4^2)+\lambda^6]/[2^{(2)(3)}(3!)^2$$

$$a_4 = \quad [\,\lambda^4 2^2 6^2+\lambda^6(2^2+4^2+6^2)+\lambda^8]/[2^{(2)(4)}(4!)^2]$$

$$a_5 = -\{\,\lambda^6[\underline{2^2 6^2+8^2(2^2+4^2)]}+\lambda^8(2^2+4^2+6^2+8^2)+\lambda^{10}\}/[2^{(2)(5)}(5!)^2]$$

$$a_6 = \quad \{\lambda^6 2^2 6^2 10^2+\,\lambda^8[2^2 6^2+8^2(2^2+4^2)+10^2(2^2+4^2+6^2)]\,+\lambda^{10}(2^2+4^2+...+10^2)+$$
$$+\lambda^{12}\}/[2^{(2)(6)}(6!)^2]$$

$$a_7 = -\{\,\lambda^8[2^2 6^2 10^2+12^2[2^2 6^2+8^2(2^2+4^2)]+\lambda^{10}[2^2 6^2+8^2(2^2+4^2)+10^2(2^2+4^2+6^2)+$$
$$+12^2(2^2+4^2+6^2+8^2)]\,+\lambda^{12}(2^2+4^2+...+12^2)+\lambda^{14}\}/[2^{(2)(7)}(7!)^2]$$

$$a_8 = \quad \{\,\lambda^8 2^2 6^2 10^2 14^2+\lambda^{10}\underline{[2^2 6^2 10^2+12^2[2^2 6^2+8^2(2^2+4^2)]+14^2[2^2 6^2+8^2(2^2+4^2)+}$$
$$\underline{+10^2(2^2+4^2+6^2)]]}+\,\lambda^{12}[2^2 6^2+8^2(2^2+4^2)+10^2(2^2+4^2+6^2)+12^2(2^2+4^2+6^2+8^2)+$$
$$+14^2(2^2+4^2+...+10^2)]\,+\lambda^{14}(2^2+4^2+...+14^2)+\lambda^{16}\}/[2^{(2)(8)}(8!)^2]$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad (4.7)$$

The underlines are explained in examples 1 and 2 below.

From these expansions of a_k, we can see the complexity of the expressions with large k. Both the number of terms and the constants increase with increasing k. Because the

denominator increases very rapidly, it will be extremely difficult to find the a_k's we need for evaluation of eigenvalues λ_n.

To see more clearly the relationship between a_k and d_k, let us construct a matrix from the expansions of a_k.

4.2.2 Construction of the $b_{i,j}$ Matrix

Since we need to add up the numbers related to the order of λ^2 in a_k the expansions of a_k can be rearranged as

k	1	2	3	4	5	6	7	...
i	(λ^2)	(λ^4)	(λ^6)	(λ^8)	(λ^{10})	(λ^{12})	(λ^{14})	...
1	$-\frac{1}{4}$	0	0	0	0	0	0	...
2	$\frac{4}{64}$	$\frac{1}{64}$	0	0	0	0	0	...
3	0	$\frac{-20}{2304}$	$\frac{-1}{2304}$	0	0	0	0	...
4	0	$\frac{144}{147456}$	$\frac{56}{147456}$	$\frac{1}{147456}$	0	0	0	...
5	0	0	$\frac{-1424}{14745600}$	$\frac{-120}{14745600}$	$\frac{-1}{14745600}$	0	0	....
6	0	0	$\frac{14400}{2123366400}$	$\frac{7024}{2123366400}$	$\frac{220}{2123366400}$	$\frac{1}{2123366400}$	0	...
7	0	0	0	$\frac{-219456}{416579814400}$	$\frac{-24304}{416579814400}$	$\frac{-364}{416579814400}$	$\frac{-1}{416579814400}$	...
8	0	0	0	$\frac{2822400}{106542032486400}$	$\frac{1596160}{106542032486400}$	$\frac{67424}{106542032486400}$	$\frac{560}{106542032486400}$	...
...	....	...	...	...	...	...	...	...

so a matrix $\boldsymbol{B}$ is constructed:

$$\boldsymbol{B} = \begin{bmatrix} b_{1,1} & 0 & 0 & 0 & 0 & 0 & \dots \\ b_{2,1} & b_{2,2} & 0 & 0 & 0 & 0 & \dots \\ 0 & b_{3,2} & b_{3,3} & 0 & 0 & 0 & \dots \\ 0 & b_{4,2} & b_{4,3} & b_{4,4} & 0 & 0 & \dots \\ 0 & 0 & b_{5,3} & b_{5,4} & b_{5,5} & 0 & \dots \\ 0 & 0 & b_{6,3} & b_{6,4} & b_{6,5} & b_{6,6} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

From the matrix and the expansions a_k, we observe that

$$b_{1,1} = \frac{(-1)^1}{2^{(2)(1)}(1!)^2}$$

$$b_{2,2} = \frac{(-1)^2}{2^{(2)(2)}(2!)^2}$$

$$b_{2,1} = b_{2,2}\ 2^2 = b_{2,2}\ 2^{(2)(1)}1^2 = b_{2,2}\ 2^{(2)(1)}\sum_{s_1=1}^{1}(s_1)^2$$

$$b_{3,3} = \frac{(-1)^3}{2^{(2)(3)}(3!)^2}$$

$$b_{3,2} = b_{3,3}\ (2^2 + 4^2) = b_{3,3}\ 2^{(2)(3-2)}(1^2 + 2^2) = b_{3,3}\ 2^{(2)(1)}\sum_{s_1=1}^{2}(s_1)^2$$

$$b_{4,4} = \frac{(-1)^4}{2^{(2)(4)}(4!)^2}$$

$$b_{4,3} = b_{4,4}\ (2^2 + 4^2 + 6^2) = b_{4,4}\ 2^{(2)(4-3)}(1^2 + 2^2 + 3^2)$$

$$= b_{4,4}\ 2^{(2)(1)}(1^2 + 2^2 + 3^2) = b_{4,4}\ 2^{(2)(1)}\sum_{s_1=1}^{3}(s_1)^2$$

$$b_{4,2} = b_{4,4}\ (2^2 6^2) = b_{4,4}\ 2^{(2)(4-2)}(3^2 1^2) = b_{4,4}\ 2^{(2)(2)}\left[\sum_{s_2=1}^{2}(s_2 + 1)^2\left[\sum_{s_1=1}^{s_2-1}(s_1)^2\right]\right]$$

....

where we define the summation for the upper limit of zero as follows:

$$\sum_{s_1=1}^{0} (s_1)^2 = 0$$

From these observations, we can deduce a formulation to calculate $b_{i,k}$ directly related to the index instead of from the expansion of each term a_k. The formulation is:

$$b_{i,i} = \frac{(-1)^i}{2^{2i}(i!)^2}$$

and for $i > k$

$$b_{i,k} = b_{i,i}\ 2^{2\Delta} \sum_{s_\Delta=\Delta}^{k} (s_\Delta + \Delta - 1)^2 [\sum_{s_{\Delta-1}=\Delta-1}^{s_\Delta - 1} (s_{\Delta-1} + \Delta - 2)^2 [\ldots[\sum_{s_2=2}^{s_3-1} (s_2 + 1)^2 [\sum_{s_1=1}^{s_2-1} s_1{}^2]]\ldots]] \tag{4.8}$$

where $\Delta = i - k$, and then the coefficients in the eigenfunction can be found from

$$d_k = \sum_{i=k}^{2k} b_{i,k} \tag{4.9}$$

Example 1:

$$b_{5,3} = b_{5,5}\ 2^{2(5-3)} [\sum_{s_2=2}^{3} (s_2 + 1)^2 [\sum_{s_1=1}^{s_2-1} s_1{}^2]\ ]$$

$$= \frac{(-1)^5}{2^{2(5)}(5!)^2}\ 2^4 [3^2 \sum_{s_1=1}^{1} s_1{}^2 + 4^2 \sum_{s_1=1}^{2} s_1{}^2]$$

$$= \frac{-1}{14,745,600}\ 2^4 [3^2 1^2 + 4^2 (1^2 + 2^2)]$$

$$= \frac{-1,424}{14,745,600}$$

Example 2:

$$b_{8,5} = b_{8,8}\ 2^{2(8-5)}\ [\sum_{s_3=3}^{5}(s_3+2)^2[\sum_{s_2=2}^{s_3-1}(s_2+1)^2[\sum_{s_1=1}^{s_2-1}s_1{}^2]\ ]\ ]$$

$$= \frac{(-1)^8}{2^{2(8)}(8!)^2}\ 2^6[5^2\sum_{s_2=2}^{3-1}(s_2+1)^2[\sum_{s_1=1}^{s_2-1}s_1{}^2] + 6^2\sum_{s_2=2}^{4-1}(s_2+1)^2[\sum_{s_1=1}^{s_2-1}s_1{}^2] +$$

$$+ 7^2\sum_{s_2=2}^{5-1}(s_2+1)^2[\sum_{s_1=1}^{s_2-1}s_1{}^2]\ ]$$

$$= \frac{1}{2^{10}(8!)^2}\ 2^6\{5^2 3^2 1^2 + 6^2[3^2 1^2 + 4^2(1^2+2^2)] +$$

$$+ 7^2[3^2 1^2 + 4^2(1^2+2^2) + 5^2(1^2+2^2+3^2)]\}$$

$$= \frac{1,596,160}{106,542,032,486,400}$$

These two examples are also shown as the underlined parts in expansion, Eq. (4.7); when applying the recursion relationship, Eq. (4.5) and the procedures described in part 4.1, the computations are not easy but rather are complex and time–consuming. For example, the computation of $b_{8,5}$ involves the expansions of $a_1,$, a_2, ..., a_8 and addition of all term with λ^{10} in those expansions. Thus, these examples also show that the computation by the formulation in Eq. (4.8) is much more effective than that found by the recursion relationship, Eq. (4.5).

4.2.3 Expansion of a_k ($Kn > 0$)

From the above two examples, we can see the correctness and effectiveness of Eq. (4.8) and, based on it, the formulation for $Kn > 0$ also can be derived.

For $\beta > 1$ the corresponding expansion of a_k are as follows:

$$a_0 = 1$$

$$a_1 = -\lambda^2\beta/[2^{(2)(1)}(1!)^2]$$

$$a_2 = [\lambda^2 2^2+\lambda^4\beta^2]/[2^{(2)(2)}(2!)^2]$$

$$a_3 = -[\lambda^4\beta(2^2+4^2)+\lambda^6\beta^3]/[2^{(2)(3)}(3!)^2]$$

$$a_4 = [\lambda^4 2^2 6^2+\lambda^6\beta^2(2^2+4^2+6^2)+\lambda^8\beta^4]/[2^{(2)(4)}(4!)^2]$$

$$a_5 = -\{\lambda^6\beta[2^2 6^2+8^2(2^2+4^2)]+\lambda^8\beta^3(2^2+4^2+6^2+8^2)+\lambda^{10}\beta^5\}/[2^{(2)(5)}(5!)^2]$$

$$a_6 = \{\lambda^6 2^2 6^2 10^2+\lambda^8\beta^2[2^2 6^2+8^2(2^2+4^2)+10^2(2^2+4^2+6^2)]+\lambda^{10}\beta^4(2^2+4^2+\ldots+10^2) +\lambda^{12}\beta^6\}/[2^{(2)(6)}(6!)^2]$$

$$a_7 = -\{\lambda^8\beta[2^2 6^2 10^2+12^2[2^2 6^2 + 8^2(2^2+4^2)] + \lambda^{10}\beta^3[2^2 6^2+8^2(2^2+4^2)+ +10^2(2^2+4^2+6^2)+12^2(2^2+4^2+6^2+8^2)] +\lambda^{12}\beta^5(2^2+4^2+\ldots+12^2)+ +\lambda^{14}\beta^7\}/[2^{(2)(7)}(7!)^2]$$

$$a_8 = \{\lambda^8 2^2 6^2 10^2 14^2+\lambda^{10}\beta^2[2^2 6^2 10^2+12^2[2^2 6^2 + 8^2(2^2+4^2)]+ +14^2[2^2 6^2+ 8^2(2^2+4^2)+10^2(2^2+4^2+6^2)]]+ \lambda^{12}\beta^4[2^2 6^2+8^2(2^2+4^2)+ +10^2(2^2+4^2+6^2)+12^2(2^2+4^2+6^2+8^2)+14^2(2^2+4^2+\ldots+10^2)]+ +\lambda^{14}\beta^6(2^2+4^2+\ldots+14^2)+\lambda^{16}\beta^8\}/[2^{(2)(8)}(8!)^2]$$

$$\cdots\cdots \qquad (4.10)$$

so the matrix is modified as follows:

$$\boldsymbol{B}' = \begin{bmatrix} b_{1,1}\beta & 0 & 0 & 0 & 0 & 0 & \dots \\ b_{2,1} & b_{2,2}\beta^2 & 0 & 0 & 0 & 0 & \dots \\ 0 & b_{3,2}\beta & b_{3,3}\beta^3 & 0 & 0 & 0 & \dots \\ 0 & b_{4,2} & b_{4,3}\beta^2 & b_{4,4}\beta^4 & 0 & 0 & \dots \\ 0 & 0 & b_{5,3}\beta & b_{5,4}\beta^3 & b_{5,5}\beta^5 & 0 & \dots \\ 0 & 0 & b_{6,3} & b_{6,4}\beta^2 & b_{6,5}\beta^4 & b_{6,6}\beta^6 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

Let

$$b_{i,k}' = b_{i,k}\beta^{i-2\Delta} \tag{4.11}$$

then

$$d_k = \sum_{i=k}^{2k} b_{i,k}' \tag{4.12}$$

All the expressions for evaluation of eigenvalues of the Graetz Problem in slip–flow are summed as follows:

$$G(r^*)\mid_{r^*=1} = G(1,\lambda) = \sum_{k=0}^{\infty} d_k\lambda^{2k} = 1 + d_1\lambda^2 + d_2\lambda^4 + d_3\lambda^6 + \dots = 0 \tag{4.6}$$

$$b_{i,i} = \frac{(-1)^i}{2^{2i}(i!)^2}$$

$$b_{i,k} = b_{i,i}\, 2^{2\Delta} \sum_{s_\Delta=\Delta}^{k} (s_\Delta + \Delta - 1)^2 [\sum_{s_{\Delta-1}=\Delta-1}^{s_\Delta - 1} (s_{\Delta-1} + \Delta - 2)^2 [\dots[\sum_{s_2=2}^{s_3-1} (s_2 + 1)^2 [\sum_{s_1=1}^{s_2-1} s_1^{\,2}]]\dots]] \tag{4.8}$$

$$b_{i,k}' = b_{i,k}\beta^{i-2\Delta} \tag{4.11}$$

$$d_k = \sum_{i=k}^{2k} b_{i,k}' \tag{4.12}$$

where $\Delta = i - k$.

From these expressions one can see that the summation notation greatly simplifies the evaluation of eigenvalues.

4.3. Summary

In this chapter, a technique for calculation of the eigenvalues occurring in the Graetz Problem in slip–flow has been derived by constructing a matrix. The two examples show that the computation of the $b_{i,k}$ by the formulation in Eq. (4.8) is much more effective than when found by the recursion relationship, Eq. (4.5). With the formulation, any number of eigenvalues can be theoretically determined. The next chapter will deal with the algorithms, the computational program, and will carry out the calculation of the eigenvalues.

CHAPTER 5

COMPUTATIONAL RESULTS

In the previous chapter, the formulation for the calculation of the coefficients occurring in the eigenfunction was determined. In this chapter, we will develop the codes for the evaluation of eigenvalues for the Graetz Problem in slip–flow. We will calculate the eigenvalues and discuss the computational results.

5.1 Treatment of Very Large Numbers in the Computations

By using Eq. (4.8), the calculation of the coefficient $b_{i,k}$ involves the constant $2^{2i}(i!)^2$, which can become a very large number. For example, if $i = 30$ (or $k = 15$), then $2^{2i}(i!)^2 = 2^{60}(30!)^2 = (1.15 \times 10^{18})(7.04 \times 10^{64}) = 8.11 \times 10^{82}$, which will overflow on the computer. Therefore, to reduce the effect of very large numbers in the computation, the eigenfunction must be treated as follows.

(1) We can combine 2^{2i} into the summation in Eq. (4.8) as

$$b_{i,k} = \frac{(-1)^i}{2^{2i}(i!)^2}\ 2^{2\Delta} \sum_{s_\Delta = \Delta}^{k} (s_\Delta + \Delta-1)^2 [\sum_{s_{\Delta-1} = \Delta-1}^{s_\Delta - 1} (s_{\Delta-1} + \Delta-2)^2 [\ldots[\sum_{s_2 = 2}^{s_3-1} (s_2 + 1)^2 [\sum_{s_1 = 1}^{s_2-1} s_1^{\,2}]]\ldots]]$$

$$= \frac{(-1)^i}{2^{2i}(i!)^2}\ 2^{4\Delta} \sum_{s_\Delta = \Delta}^{k} \frac{(s_\Delta + \Delta-1)^2}{2^2} [\sum_{s_{\Delta-1} = \Delta-1}^{s_\Delta - 1} \frac{(s_{\Delta-1} + \Delta-2)^2}{2^2} [\ldots[\sum_{s_2 = 2}^{s_3-1} \frac{(s_2 + 1)^2}{2^2} [\sum_{s_1 = 1}^{s_2-1} \frac{s_1^{\,2}}{2^2}]]\ldots]]$$

$$= \frac{(-1)^i}{2^{4k-2i}(i!)^2} \sum_{s_\Delta = \Delta}^{k} \frac{(s_\Delta + \Delta-1)^2}{2^2} [\sum_{s_{\Delta-1} = \Delta-1}^{s_\Delta - 1} \frac{(s_{\Delta-1} + \Delta-2)^2}{2^2} [\ldots[\sum_{s_2 = 2}^{s_3-1} \frac{(s_2 + 1)^2}{2^2} [\sum_{s_1 = 1}^{s_2-1} \frac{s_1^{\,2}}{2^2}]]\ldots]]$$

So for $i = 30$ (or $k = 15$), the largest number in the term $b_{2k,k}$ can be reduced to $(30!)^2 = 7.04 \times 10^{64}$.

(2) Letting $\lambda' = \lambda/g$ and $d_k' = g^{2k}d_k$ and allowing the magnification coefficient, g, be any given number greater than 1, we have

$$G(\lambda) = \sum_{k=0}^{\infty} \lambda^{2k} d_k = 1 + \lambda^2 d_1 + \lambda^4 d_2 + \lambda^6 d_3 + \ldots$$

$$= 1 + (\lambda' g)^2 \frac{d_1'}{g^2} + (\lambda' g)^4 \frac{d_2'}{g^4} + (\lambda' g)^6 \frac{d_3'}{g^6} + \ldots = \sum_{k=0}^{\infty} (\lambda' g)^{2k} \frac{d_k'}{g^{2k}}$$

$$= 1 + \lambda'^2 d_1' + \lambda'^4 d_2' + \lambda'^6 d_3' + \ldots$$

$$= \sum_{k=0}^{\infty} \lambda'^{2k} d_k' = G(\lambda') = 0 \qquad (5.1)$$

For example, when we let g be 10, then from Eq. (4.9) and Eq. (4.8) we have

$$d_1' = g^{2(1)} d_1 = 10^{2(1)} d_1 = 10^{2(1)} \sum_{i=1}^{2(1)} b_{i,1} = \sum_{i=1}^{2(1)} 10^{2(1)} b_{i,1}$$

$$= 10^{2(1)} b_{1,1} + 10^{2(1)} b_{2,1}$$

$$d_2' = g^{2(2)} d_2 = 10^{2(2)} d_2 = 10^{2(2)} \sum_{i=2}^{2(2)} b_{i,2} = \sum_{i=2}^{2(2)} 10^{2(2)} b_{i,2}$$

$$= 10^{2(2)} b_{2,2} + 10^{2(2)} b_{3,2} + 10^{2(2)} b_{4,2}$$

$$d_3' = g^{2(3)} d_3 = 10^{2(3)} d_3 = 10^{2(3)} \sum_{i=3}^{2(3)} b_{i,3} = \sum_{i=3}^{2(3)} 10^{2(3)} b_{i,3}$$

$$= 10^{2(3)} b_{3,3} + 10^{2(3)} b_{4,3} + 10^{2(3)} b_{5,3} + 10^{2(3)} b_{6,3}$$

....

In general then,

$$d_k' = g^{2k} d_k = g^{2k} \sum_{i=k}^{2k} b_{i,k} = \sum_{i=k}^{2k} g^{2k} b_{i,k}$$

....

For the example in which $k = 15$, we have

$$d_{15}' = 10^{2(15)} d_{15} = 10^{2(15)} \sum_{i=15}^{2(15)} b_{i,15} = \sum_{i=15}^{2(15)} 10^{2(15)} b_{i,15}$$

Thus, combining $10^{2(15)}$ into the term $b_{30,15}$, the largest number in this term reduces to $(i!)^2/10^{2k} = (30!)^2/10^{30} = 7.04 \times 10^{34}$.

(3) The magnitude of the term $|b_{i,i}|$ can be reduced for computational purposes by taking the logarithm of both sides and later reversing this computation by taking the exponential function of both sides. That is,

$$log_{10} b_{i,i} = -2i \, log_{10} 2 - 2 \sum_{k=1}^{i} \log_{10} k \qquad (5.2)$$

This method can reduce the number greatly. For example, for i = 30, then,

$$log_{10} b_{30,30} = -2(30) \, log_{10} 2 - 2 \sum_{k=1}^{30} \log_{10} k$$

$$= -(60)\ (0.301) - 64.847 = -82.907$$

All three of these methods were used in the computer code to accommodate the inherently large numbers.

5.2 Flow Chart of Computation

A block diagram for computation of the eigenvalues is shown in Figure 5.1. The input data includes the number of terms, k, the magnification coefficient, g, in Eq. (5.1), and the Knudsen number, Kn or β in Eq. (4.11).

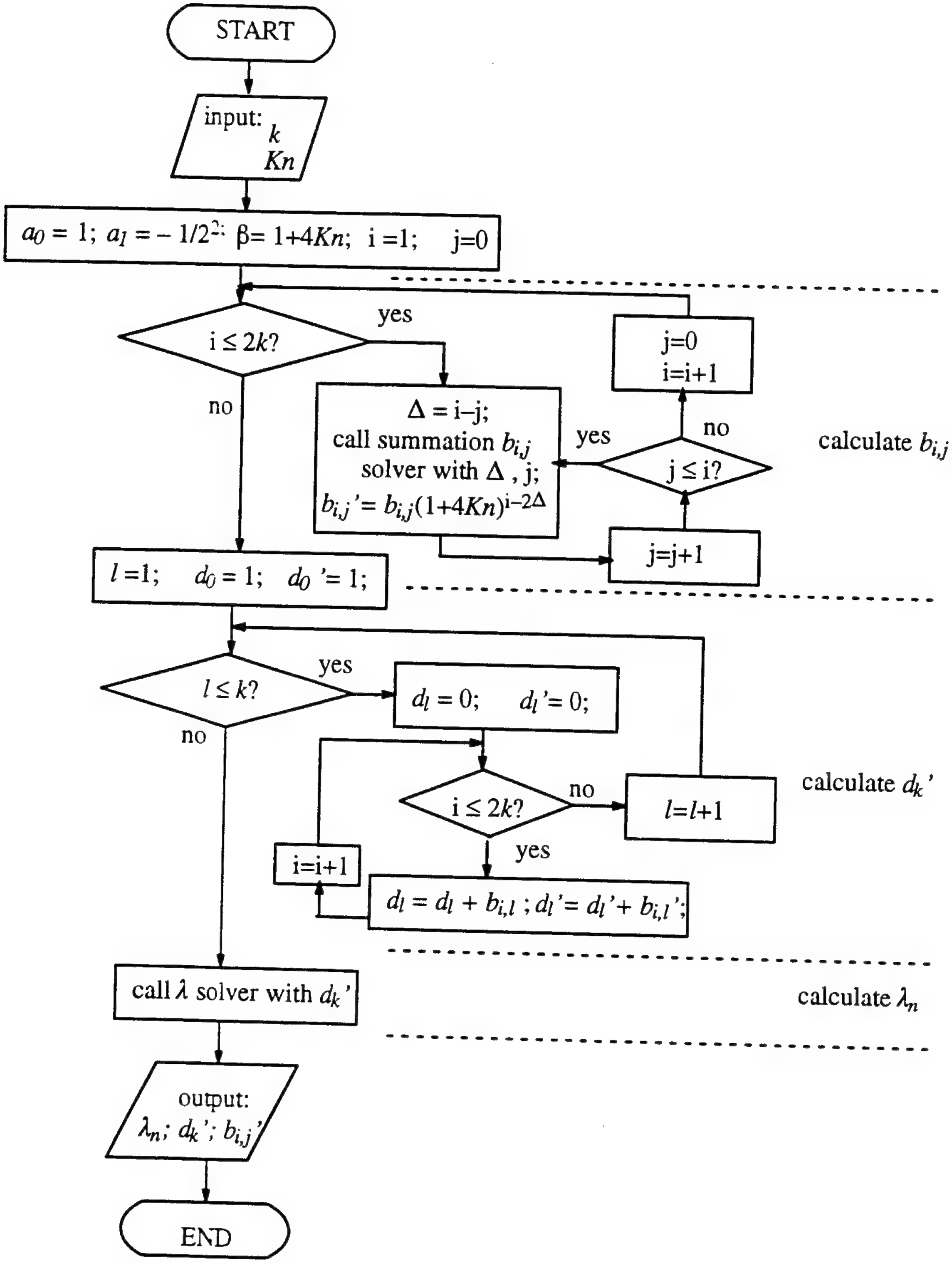

Fig. 5.1 Flow chart of computation of eigenvalues

The calculation procedure can be broken down into three steps:

1) calculation of coefficient $b_{i,j}$ (which requires a summation solver for Eq. (4.8) for $b_{i,j}$ and Eq. (4.11) for $b_{i,j}'$);

2) calculation of d_k by Eq. (4.9) and d_k' by Eq. (4.12); and

3) calculation of eigenvalues λ_n, or λ_n' by Eq. (5.1).

Based on the flow chart, the codes for computation of the eigenvalues have been developed and are listed in Appendix A. The codes need as input the value of the Knudsen number, *Kn*, and the number of coefficients d_k, or *k*, and the magnification coefficient, *g*. The output of the calculation includes the coefficient $b_{i,j}$, d_k, and eigenvalues, λ_n.

From Examples 1 and 2 in Chapter 4 for the formulation of Eq. (4.8), we can see the feature of the calculation of the *k*–th term: the calculation involves all the previous (*k*–1) terms, that is, the summations of parameters $s_1, s_2, \ldots, s_{k-1}$, and further expanded summations of $s_1, s_2, \ldots, s_{k-1}$, for the present *k*. This concept can be illustrated using the following triangles in Fig. 5.2.

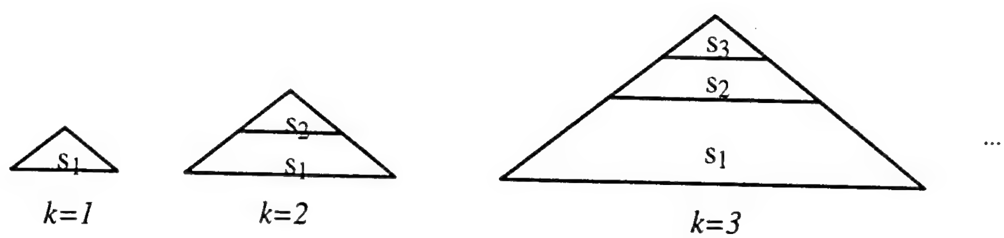

Fig. 5.2 Illustration of the growth of $s_1, s_2, \ldots, s_{k-1}$ as functions of *k*

5.3 Results

5.3.1 Comparison of the First Ten d_k ($Kn = 0$)

5.3.1.1 Accuracy. Because the evaluation of eigenvalues involves the determination of an infinite number of terms, it is extremely difficult to determine directly the accuracy of

the approximate numerical eigenvalues. All we have to do is to find the consistencies existing in the results by different methods instead to determine the relative accuracy.

In order to assess the accuracy of the computation of Eq. (4.8), a comparison of the first ten d_k using Mathcad 5.0 was made, as shown in Table 5.1. In the table, the data in the second column are the computational coefficients d_k (with $g = 10$) using Eq. (4.8); the data in the fourth column are the absolutely exact coefficients d_k calculated by expansions using the symbolic processor in Mathcad 5.0; the third column gives the equivalent decimal values of the fourth column. The differences between Eq. (4.8) and the exact values are shown in Table 5.2.

Table 5.1 Comparison of Coefficients d_k (g=10)

k	Eq. (4.9)	Simulation by Mathcad Version 5.0	
		Numeric equivalent	Symbolic solution
0	1.0000	1.00000	1
1	−18.7500	−18.75000E−02	−3/16
2	79.2101	79.210069E−04	73/9,216
3	−144.043	−144.04297E−06	−59/409,600
4	145.080	145.07980E−08	603,793/416,197,814,400
5	−92.6715	−92.67144E−10	−555,379/59,929,893,273,600
6	40.8619	40.861856E−12	4,266,870,481/104,421,846,039,920,640,000
7	−13.1812	−13.181156E−14	−37,217,872,147/282,356,671,691,945,410,560,000
8	3.24941	3.2451315E−16	41,377,942,693,441/127,507,755,229,335,476,282,327,040,000
9	−0.646315	−0.62971547E−18	−9,281,940,782,645,851/14,739,896,504,511,181,058,237,005,824E+6

Table 5.2 Differences of d_k Between Two Methods

k	0	1	2	3	4	5	6	7	8	9
Difference	0	1	3.1E–9	3.0E–11	2.0E–12	5.5E–15	1.4E–17	4.4E–19	4.3E–19	1.7E–20

From the comparison, we can see that the coefficient d_k computed by Eq. (4.8) is always a little larger than that found by Mathcad, but the differences are rather small. Based on the comparison, it is evident that the computational results have enough accuracy for an accurate determination of the eigenvalues.

5.3.1.2 Efficiency. In order to assess the efficiency of the computation of Eq. (4.8), we can compare the computational efficiencies using Mathcad and a calculation of Eq. (4.8) using a Fortran code for $k = 9$.

Using Mathcad 5.0, three steps were implemented to solve for the eigenvalues:

1) Expand $a_0, a_1, a_2, \ldots, a_{18}$, as shown in Chapter 4;
2) Add up the terms in the expansions with respect to λ^2 for d_1, λ^4 for $d_2, \ldots$, λ^{18} for d_9; and
3) Solve the eigenfunction equation for the eigenvalues

$$1 + d_1\lambda^2 + d_2\lambda^4 + d_3\lambda^6 + \ldots + d_9\lambda^{18} = 0$$

Procedure 1) took about ten hours and 2) about five hours to accomplish. For large k, for example $k = 25$, it would be too time–consuming and complex to use those methods.

Using the computer code given in Appendix A, which uses Eq. (4.8) to determine all d_k for $k = 1$ to 9, the process took approximately one minute. Therefore, the latter procedure is an effective and efficient technique for the calculation of eigenvalues.

5.3.2 Behavior of the Eigenfunction

In practice, the number of terms used in the eigenfunction expression must be limited. In order to see how the eigenfunction behaves with respect to the number of coefficient terms, a number of calculations have been carried out and plotted.

Figure 5.3 shows the behavior of the eigenfunction as a function of the number of coefficient terms. The numbers of the curves indicate the number of terms d_k used in the computations. At least six terms are needed to obtain the two lowest eigenvalues, shown as the curve numbered 6.

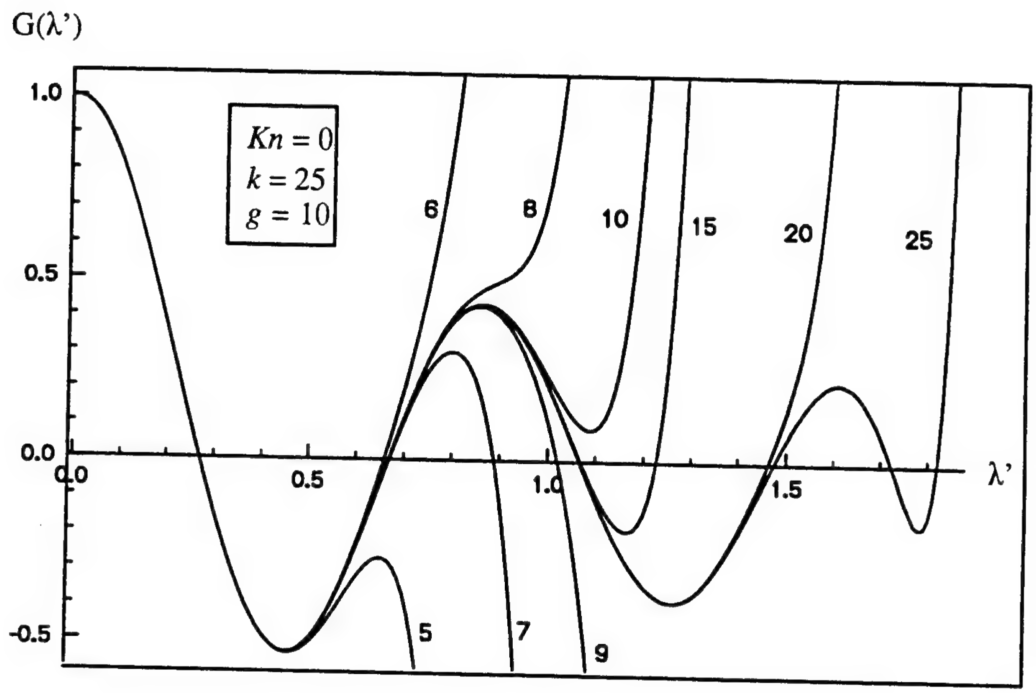

Fig.5.3 Behavior of the eigenfunction as the number of coefficient terms increases

Graetz found the first two values, $\lambda_1 = 2.7043$ and $\lambda_2 = 6.50$ from the following eigenfunction:

$$0 = 1 - \lambda^2 0.1875 + \lambda^4 0.007921 - \lambda^6 0.00014404 + \lambda^8 145.92\text{x}10^{-8} + - \lambda^{10} 94.938\text{x}10^{-10} + \ldots \quad (5.3)$$

Comparing the coefficients d_k from Eq. (5.3) with those given in Table 5.1 we can see significant differences exist from the 5th term on, revealing why Graetz's second eigenvalue is not accurate.

Obviously, the second eigenvalue determined by curve 6 is less than the "accurate value," which is slightly greater than 6.6 (see Table 5.3), shown as curves 7, 8, and 9 etc., in Figure 5.4.

In Figure 5.4, we can see that the true second eigenvalue lies between those obtained by curves 7 and 8. It is clear that the second eigenvalue obtained initially by curve 6 is rather rough; with one more term, the value is larger than the first one as shown by curve 7, but closer to the "accurate value"; the second eigenvalue given by curve 8 is a little less than that given by curve 7 but is even closer to the "accurate value." The same holds true for the case with nine terms, ten terms, and so on.

From this discussion we can conclude:

(1) that the eigenvalues initially obtained with the minimum number of coeffi–cients is always rather rough, such as the second eigenvalue obtained by curve 6 and the third eigenvalue obtained by curve 9;

(2) that the next to the last available eigenvalue that can be determined for a certain number of coefficients is always correct and sufficiently accurate. For instance, for curve 7, the second eigenvalue can be assumed to be reasonably accurate and correct; and

(3) that the eigenvalues and the convergence of the eigenfunction are sensitive to the accuracy of the coefficients d_k.

Figure 5.5 shows the plot of the eigenfunction with 25 terms. There are six eigenvalues shown in the plot. From the above discussion, we conclude that the first five eigenvalues are correct, but the last one, or the sixth, is somewhat inaccurate due to the truncation

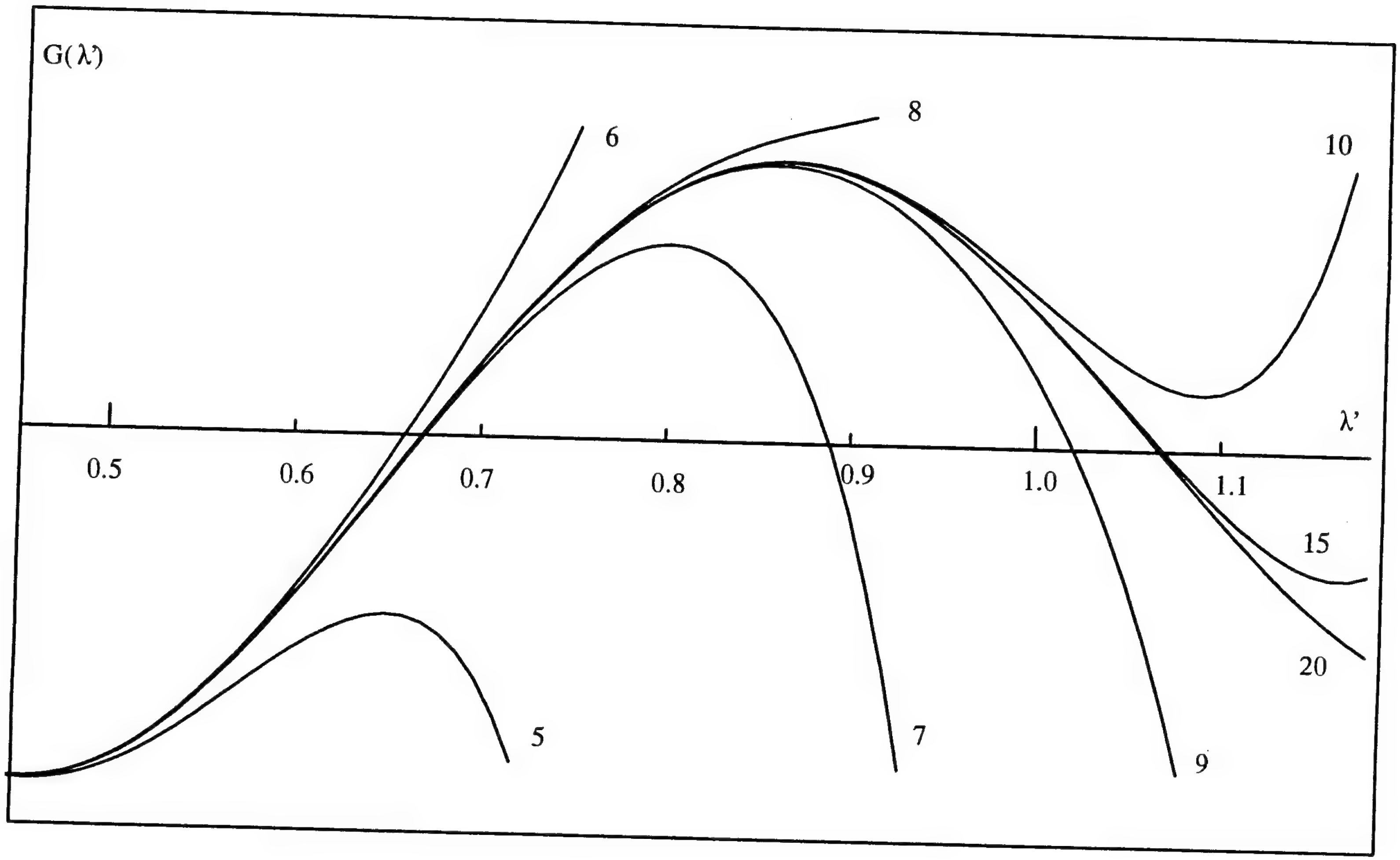

Fig. 5.4 Second eigenvalue as a function of the number of coefficients

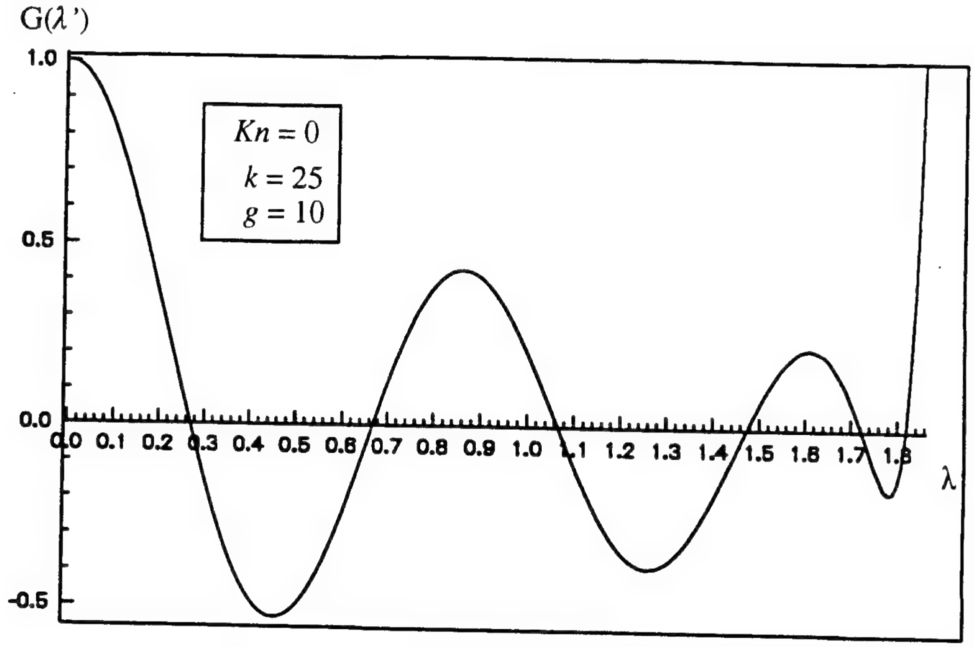

Fig. 5.5 Plot of eigenfunction with 25 coefficient terms

of the eigenfunction expression. The plot also shows that the eigenfunction is oscillating with decreasing magnitude.

5.3.3 Comparison With Previously Known Eigenvalues With $Kn = 0$

Table 5.3 shows the comparison with previously known eigenvalues for the classical "Graetz Problem" ($\beta = 1$ or $Kn = 0$) as presented by Sellars et al. (1956). As seen in Table 5.3, the first four eigenvalues are in excellent agreement. Therefore, we can apply this technique to the Graetz Problem in slip–flow for evaluation of the eigenvalues.

5.3.4 Eigenvalues for $Kn > 0$

Table 5.4 shows the first five eigenvalues for slip–flow with different Knudsen numbers, Kn.

Table 5.3 Comparison with Previously Known Eigenvalues

	Sellar et al.	Jakob	Analog computer	Present paper
n	λ_n	λ_n	λ_n	λ_n
1	2.667	2.705	2.71	2.704
2	6.667	6.66	6.69	6.679
3	10.667	10.3	10.62	10.670
4	14.667	14.67	14.58	14.761
5	18.667			17.255

Table 5.4 Eigenvalues for Different *Kn*

Kn	λ_1	λ_2	λ_3	λ_4	λ_5
0.00	2.704	6.679	10.670	14.761	17.255
0.005	2.671	6.584	10.512	14.220	
0.01	2.639	6.493	10.359	14.209	
0.02	2.578	6.320	10.071	13.815	16.576
0.04	2.468	6.013	9.561	13.099	15.836
0.06	2.371	5.747	9.120	12.560	14.646
0.08	2.284	5.513	8.737	11.963	14.573
0.10	2.206	5.305	8.396	11.514	13.938
0.12	2.136	5.119	8.096	11.074	14.273

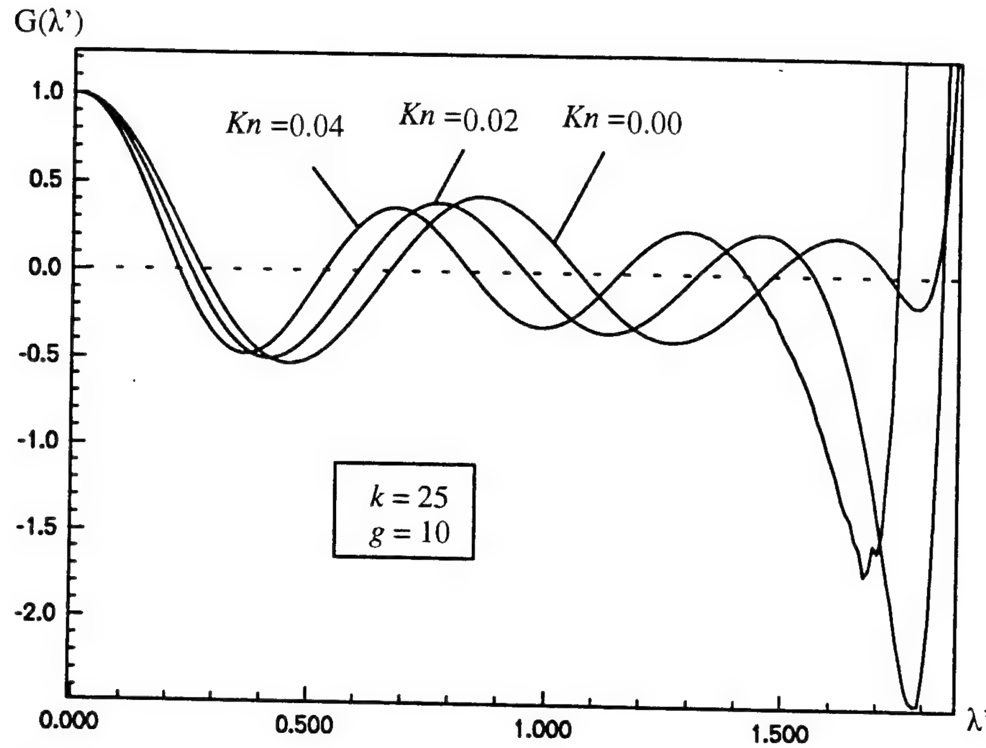

Figure 5.6 Plots of the eigenfunction as a function of Knudsen number

Figure 5.6 shows the behavior of the eigenfunction for various *Kn* under slip–flow conditions. It shows that the eigenvalues decrease as *Kn* increases. For $Kn > 0$, the plots appear unstable after the fifth root so that only the first four values are reliable. The possible cause for this instability is that the truncation errors are magnified by the factor $(1+4Kn)^i$ on $b_{i,i}$ in the modified matrix $\boldsymbol{B}$. The coefficients $d_k{}'$ for *Kn* from 0.00 to 0.12 are shown in Appendix B.

5.3.5 Influence of *Kn* on the Nusselt Number

5.3.5.1 Local heat transfer coefficient. Using Eq. (3.29) and Eq. (3.20), the local heat transfer coefficient Nu_x has been calculated.

Figure 5.7 shows the local Nu_x value as a function of x*/Gz for *Kn* = 0.02 and with the number of eigenvalues as a parameter. The value of the local Nusselt number converges dramatically with the increase in the number of eigenvalues in the computation. When x*/Gz is 0.02, the error in Nu_x is 0.7 percent when two eigenvalues are used and comparing to the straight line (using one eigenvalue), the error is 14 percent. It can be concluded that the results using four eigenvalues are sufficiently accurate for x*/Gz > 0.02. When x*/Gz is greater than 0.05, the error is at most 1.3 percent – that is, all three plots become nearly flat, indicating a thermally fully–developed condition.

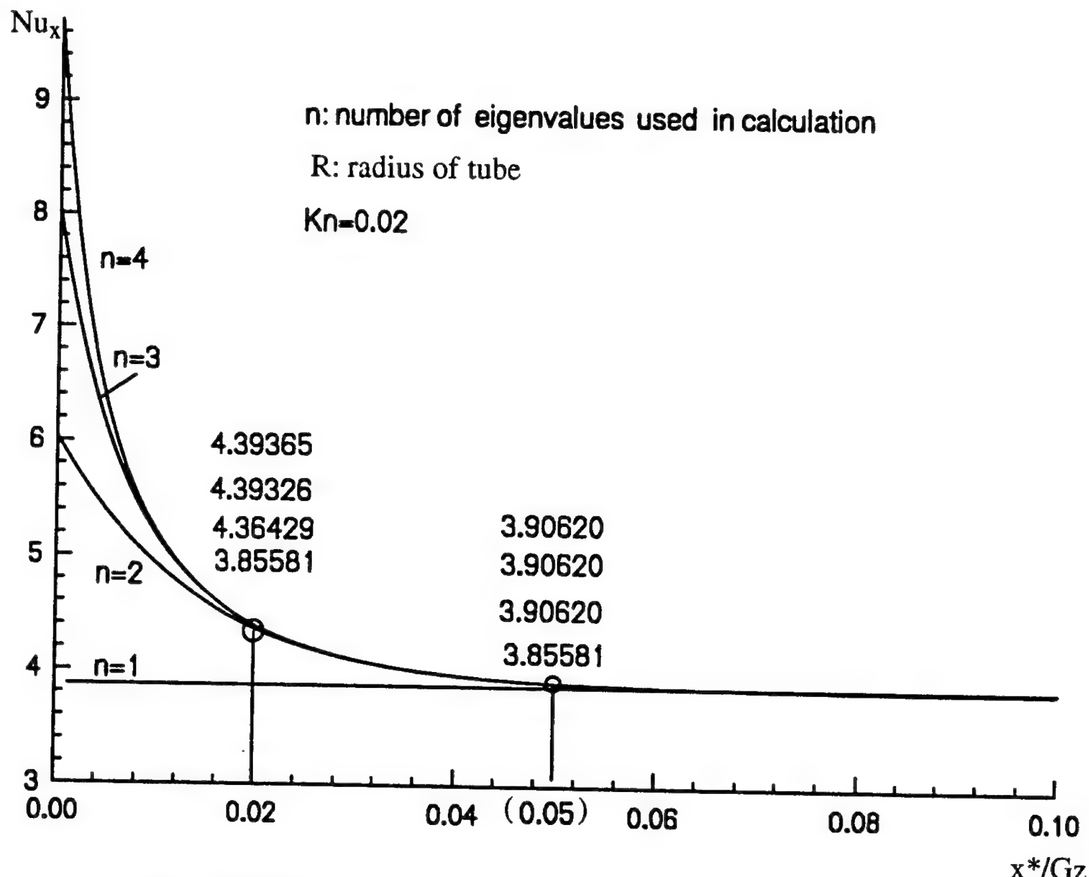

Fig. 5.7 The local Nusselt number As a function of x*/Gz

Figure 5.8 shows the local Nusselt numbers as a function of *Kn*. It is obvious that *Kn* has an influence on the Nusselt number. All the plots in Fig. 5.8 show that the Nusselt

number increases as *Kn* increases, and that this effect is magnified near the entrance. When x*/Gz is greater than 0.05, all the plots become nearly flat, indicating a thermally fully–developed condition.

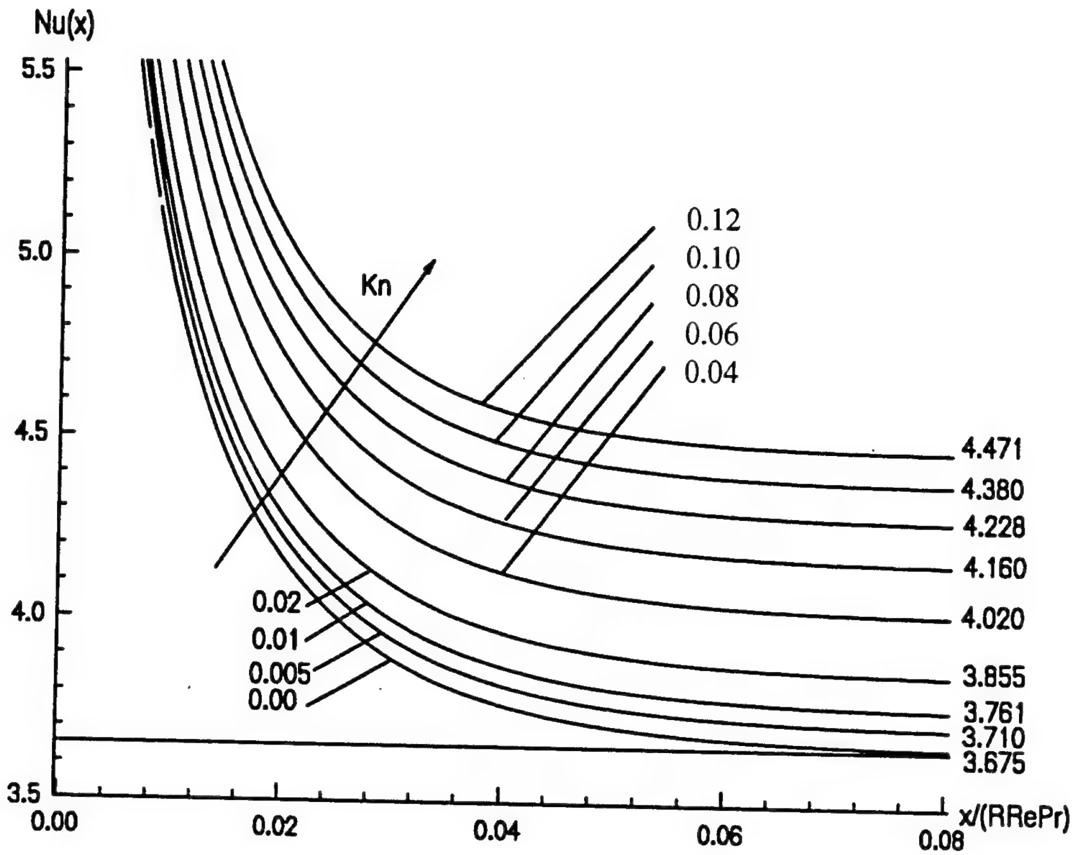

Fig. 5.8 The local Nusselt numbers as functions of x*/Gz and *Kn* for n = 4

Table 5.5 shows the values of the Nusselt number for fully–developed conditions (where x*/Gz > 0.05) for different *Kn* and the ratios of these values to those with *Kn* = 0. This ratio increases with an increase of *Kn*. The data show that when *Kn* is 0.01, the value of the Nusselt number increases about 3 percent, and when *Kn* is 0.02, the increase is greater than 5 percent. Thus, we can conclude that when *Kn* is greater than 0.01, the effect

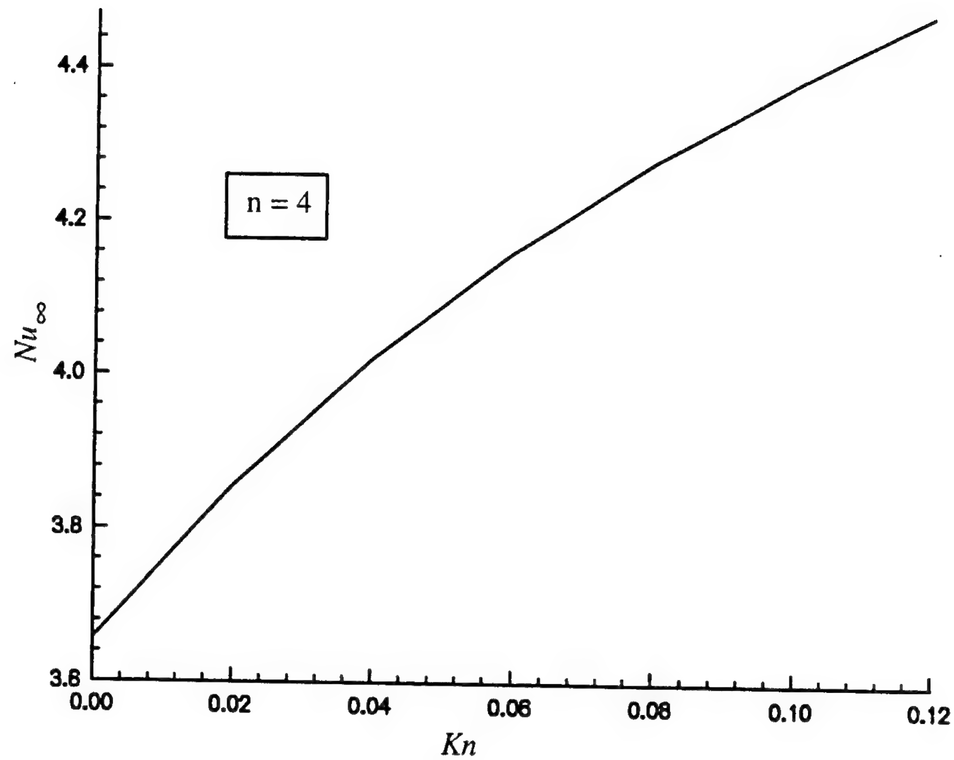

Fig. 5.9 Fully developed *Nu* as a function of *Kn*

<u>**5.3.5.2 Overall heat transfer coefficient**</u>. Using Eqs. (3.37), (3.34), and (3.39), the overall heat transfer coefficient $\overline{Nu}$ can be calculated. Fig. 5.10 shows the plots of $\overline{Nu}$, Nu_x, and ($\overline{Nu} - Nu\infty$), or Δ as functions of x*/Gz.

From Fig. 5.10, we can see that the overall heat transfer coefficient $\overline{Nu}$ is greater than the local heat transfer coefficient Nu_x: that the entrance has a greater effect on $\overline{Nu}$ than on Nu_x: and that when x*/Gz is greater than 0.05, Nu_x becomes nearly flat, indicating a thermally fully–developed condition. However, $\overline{Nu}$ does not becomes flat, so that it can not be considered as a fully–developed condition until x*/Gz greater than 0.20.

From the above discussion, we can see that:

(1) slip–flow has a positive influence on the heat transfer coefficient and can enhance the heat transfer efficiency;

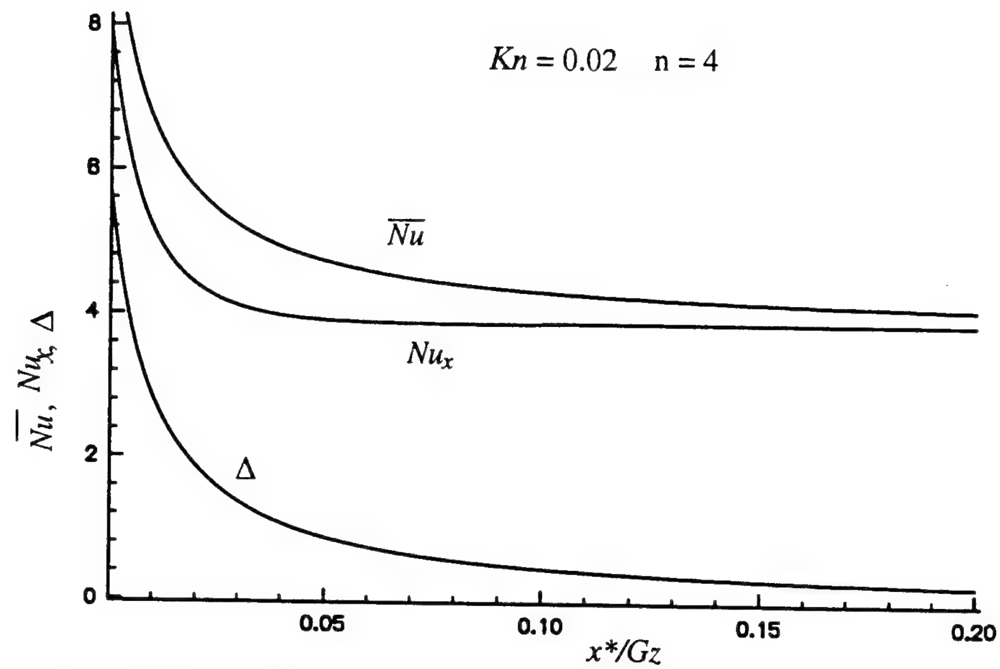

Fig. 5.10 Plots of $\overline{\text{Nu}}$, Nu_x and Δ as functions of dimensionless axial location

(2) the influence depends on the Knudsen number and increases as *Kn* increases;

(3) when *Kn* is equal or greater than 0.02, the increase in the fully–developed Nu is greater than 5 percent so that the effect of slip–flow should be taken into consider–ation in the computations of the heat transfer coefficient; and

(4) that the influence of *Kn* on Nu_∞ will decrease as *Kn* increases.

5.4 Simplified Eigenvalue Relationships with *Kn*

Figure 5.11 shows that λ_n are functions of *Kn*. For practical purposes, a simplified expression for calculation of the eigenvalues is needed. By using a least–squares curve fit program, the following exponential expressions as given in Table 5.4 were found to yield the best fit. The general form may be expressed as

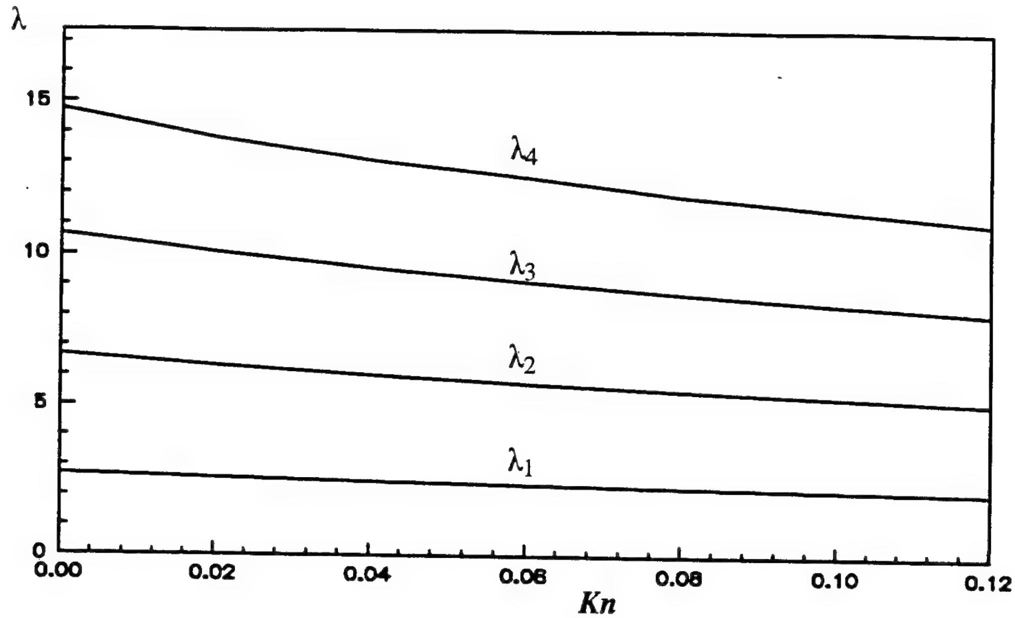

Fig.5.11 The first four eigenvalues as a function of Kn

$$\lambda_n = C_1 + C_2\, Kn \exp(C_3\, Kn) \tag{5.4}$$

The constants C_1, C_2, C_3 and the correlation coefficient R^2 are listed in Table 5.6, as functions of *Kn*.

Table 5.6 Coefficients in Eq. (5.4) as Functions of λ

n	λ_n	C_1	C_2	C_3	R^2
1	2.704	2.704	–6.6236	–2.8482	0.9997
2	6.679	6.679	–18.9118	–3.2003	0.9997
3	10.670	10.670	–31.7454	–3.3293	0.9972
4	14.761	14.761	–49.5056	–4.2066	0.9637

5.5 Summary

In this chapter, the codes were developed for the evaluation of eigenvalues. The first six eigenvalues were found, and the behavior of the eigenfunction was plotted. From the comparison and discussion, it is evident that the new technique using Eq. (4.8) for evaluation of the eigenvalues of the Graetz Problem in slip–flow is computionally effective and efficient; the Nusselt number increases as the Knudsen number increases; and the simplified relationship of the eigenvalues as a function of the Knudsen number is reliable and convenient for calculation purposes.

CHAPTER 6

CONCLUSIONS AND FURTHER RESEARCH

6.1 Conclusions

In the previous chapters, the mathematical models of velocity distribution and temperature distribution were established, and the expression for the series solution shows the importance of the eigenvalues. Since those eigenvalues were extremely difficult to evaluate directly from the original expansion, a formulation was derived based on a specially constructed matrix of coefficients $b_{i,j}$. The formulation can be used to find any given $b_{i,j}$ using only the indices i and j. This fact makes it possible to evaluate the eigenvalues by computer. The computer codes were developed and some results were obtained. From the discussions and analysis of the computational results, the following conclusions can be drawn:

1. The technique for evaluation of the eigenvalues of the Graetz Problem in slip–flow is computational effective;
2. The Nusselt number increases as Kn increases, or the heat transfer is enhanced under slip–flow conditions;
3. When Kn is equal to or greater than 0.02, the increase in fully developed Nusselt number is greater than 5 percent so that the effect of slip flow conditions should be taken into consideration in the computations of the heat transfer coeffient; and
4. The simplified relationship between the eigenvalues and the Knudsen number is reliable and convenient for calculation purposes.

6.2 Further Research

The evaluation of the eigenvalues is very important for the solution of the Graetz Problem in slip–flow. Although the technique is effective for $n < 5$, it is extremely time–consuming, and the computational error is a problem for large λ. Based on this work, the following future research is suggested:

1. Develop computer codes for calculating the heat transfer coefficient and Nusselt number, which involves the computation of C_n and differentiation of eigenfunction G_n at $r^* = 1$;
2. Obtain a simplified relationship between the overall Nusselt number and the Knudsen number;
3. Develop a more effective technique to deal with very large numbers in computa– tions;
4. Improve the codes to reduce the computing time; and/or
5. Develop a more effective technique to reduce the computing time.

APPENDIX A

PROGRAMS FOR COMPUTATION OF EIGENVALUES

```
*******************************************************************************
*          THIS PROGRAM FOR CALCULATION OF COEFFICIENT a_{i,j} and d_k        *
*          a:  a_{i,j}                                                        *
*          at:  initial of a_{i,j}                                            *
*          d: d_k                                                             *
*          g: magnification coefficient                                       *
*          kn: Knudsen number                                                 *
*          bb: sqrt(1.+4.*kn)                                                 *
*          s:  Δ = i-j                                                        *
*******************************************************************************


          real a(1000,1000),d(0:1000),r,u,at,sum,g,kn,bb
          integer s

C INPUT Kn, g
          read *,kn, g
          bb=sqrt(1.+4.*kn)

C CALCULATING a_{i,i}
          do 10 i=1,26
             a(i,i)=(-1)**i*(((g/2)**(i)/(funt1(i)))**2)*bb**(2*i)
10        continue

C CALCULATING a_{i,j}
          do 50 j=1,25
           at=1
           print *,'***',j
          do 40 i=j+1,2*j
           s=i-j
           summ(i,j,s,at)
           a(i,j)=at*(-1)**i*(((b/2)**(j)/((funt1(i)/10.**(s/2.)/(2**s))))**2)*bb**(2*i-4*s)

C OUTPUT a_{i,j}
          write(*,110)i,j,a(i,j)
110       format(i5,5x,i5,5x,e10.4)
          at=1
40        continue
*         print*,a(i,j)
50        continue

C INITIATING d_k zreo
          do 55 j=0,25
           d(j)=0
55        continue
```

```
C  CALCULATING d_k BY SUMMATION a_k,j
          d(0)=1
          do 60 j=1,25
            do 70 i=j,2*j
              d(j)=d(j)+a(i,j)
70          continue
60        continue

C  OUTPUT k,Kn,d_k
          open(3)
          write(3,*)'k = 25   Kn = ',kn
           do 100 i=0,25
            print *,d(i)
            write(3,*)d(i)
100        continue

C  CALCULATING THE VALUES OF EIGENFUNCTION AT EACH POINT
          du=0.0001
          open(3)
          do 200 i=1,30000
            u=du*(i-1)
            write(3,*)u,f(u,25,d),df(u,25,d)
200       continue
          call newton(d,25)
          end
*******************************************
C  FUNCTION OF n!
          function funt1(n)
           funt1=1
           do 10 i=1,n
            funt1=funt1*i
10        continue
          return
          end
****************************************************
*(1)      FUNCTION OF THE LOWEST SUMMATION
          function sum1(jj)
          real sum1

          sum1=0
           do 10 i=1,jj
            sum1=sum1+i**2/10.
10        continue
          sum1=sum1/4.
          return
          end
```

```
****************************************************
*(2)       FUNCTION OF THE  2ND LOWEST SUMMATION
           function sum2(jj)
           real sum2
           integer ss
           sum2=0
            do 10 ss=2,jj
             sum2=sum2+sum1(ss-1)*(ss+1)**2/10.
 10        continue
           sum2=sum2/4.
           return
           end
****************************************************
*(3)
           function sum3(jj)
           real sum3
           integer ss

           sum3=0
            do 10 ss=3,jj
             sum3=sum3+(ss+2)**2*sum2(ss-1)/10.
 10        continue
           sum3=sum3/4.
           return
           end
************************************************
*(4)
           function sum4(jj)
           real sum4
           integer ss

           sum4=0
            do 10 ss=4,jj
             sum4=sum4+(ss+3)**2*sum3(ss-1)/10.
 10        continue
           sum4=sum4/4.
           return
           end
********************
*(5)
           function sum5(jj)
           real sum5
           integer ss

           sum5=0
            do 10 ss=5,jj
             sum5=sum5+(ss+4)**2*sum4(ss-1)/10.
 10        continue
```

```
      sum5=sum5/4.
      return
      end
*****************************************************
*(6)
      function sum6(jj)
      real sum6
      integer ss
      sum6=0
       do 10 ss=6,jj
        sum6=sum6+sum5(ss-1)*(ss+5)**2/10.
 10   continue
      sum6=sum6/4.
      return
      end
*****************************************************
*(7)
      function sum7(jj)
      real sum7
      integer ss

      sum7=0
       do 10 ss=7,jj
        sum7=sum7+(ss+6)**2*sum6(ss-1)/10.
 10   continue
      sum7=sum7/4.
      return
      end
************************************************
*(8)
      function sum8(jj)
      real sum8
      integer ss

      sum8=0
       do 10 ss=8,jj
        sum8=sum8+(ss+7)**2*sum7(ss-1)/10.
 10   continue
      sum4=sum4/4.
      return
      end
*****************************************************
*(9)
      function sum9(jj)
      real sum9
      integer ss
```

```
      sum9=0
        do 10 ss=9,jj
          sum9=sum9+(ss+8)**2*sum8(ss-1)/10.
 10   continue
      sum9=sum9/4.
      return
      end
***********************************************
*(10)
      function sum10(jj)
      real sum10
      integer ss

      sum10=0
        do 10 ss=10,jj
          sum10=sum10+(ss+9)**2*sum9(ss-1)/10.
 10   continue
      sum10=sum10/4.
      return
      end
****************************************************
*(11)
      function sum11(jj)
      real sum11
      integer ss

      sum11=0
        do 10 ss=11,jj
          sum11=sum11+(ss+10)**2*sum10(ss-1)/10.
 10   continue
      sum11=sum11/4.
      return
      end
***********************************************
*(12)
      function sum12(jj)
      real sum12
      integer ss

      sum12=0
        do 10 ss=12,jj
          sum12=sum12+(ss+11)**2*sum11(ss-1)/10.
 10   continue
      sum12=sum12/4.
      return
      end
****************************************************
*(13)
```

```
      function sum13(jj)
      real sum13
      integer ss

      sum13=0
       do 10 ss=13,jj
        sum13=sum13+(ss+12)**2*sum12(ss-1)/10.
 10   continue
      sum13=sum13/4.
      return
      end
*************************************************
*(14)
      function sum14(jj)
      real sum14
      integer ss

      sum14=0
       do 10 ss=14,jj
        sum14=sum14+(ss+13)**2*sum13(ss-1)/10.
 10   continue
      sum14=sum14/4.
      return
      end
*****************************************************
*(15)
      function sum15(jj)
      real sum15
      integer ss
      sum15=0
       do 10 ss=15,jj
        sum15=sum15+(ss+14)**2*sum14(ss-1)/10.
 10   continue
      sum15=sum15/4.
      return
      end
*************************************************
*(16)
      function sum16(jj)
      real sum16
      integer ss

      sum16=0
       do 10 ss=16,jj
        sum16=sum16+(ss+15)**2*sum15(ss-1)/10.
 10   continue
      sum16=sum16/4.
      return
```

```
*************************************************
*(17)
          function sum17(jj)
          real sum17
          integer ss

          sum17=0
           do 10 ss=17,jj
            sum17=sum17+(ss+16)**2*sum16(ss-1)/10.
 10       continue
          sum17=sum17/4.
          return
          end
*************************************************
*(18)
          function sum18(jj)
          real sum18
          integer ss

          sum18=0
           do 10 ss=18,jj
            sum18=sum18+(ss+17)**2*sum17(ss-1)/10.
 10       continue
          sum18=sum18/4.
          return
          end
*************************************************
*(19)
          function sum19(jj)
          real sum19
          integer ss

          sum19=0
           do 10 ss=19,jj
            sum19=sum19+(ss+18)**2*sum18(ss-1)/10.
 10       continue
          sum19=sum19/4.
          return
          end
*************************************************
*(20)
          function sum20(jj)
          real sum20
          integer ss

          sum20=0
           do 10 ss=20,jj
            sum20=sum20+(ss+19)**2*sum19(ss-1)/10.
```

```
c          print*,'sum20=',sum20
 10        continue
           sum20=sum20/4.
           return
           end
*****************************************************
*(21)
           function sum21(jj)
           real sum21
           integer ss

           sum21=0
            do 10 ss=21,jj
             sum21=sum21+(ss+20)**2*sum20(ss-1)/10.
c          print*,'sum21=' ,sum21
10         continue
           sum21=sum21/4.
           return
           end
************************************************
*(22)
           function sum22(jj)
           real sum22
           integer ss

           sum22=0
            do 10 ss=22,jj
             sum22=sum22+(ss+21)**2*sum21(ss-1)/10.
c          print*,'sum22',sum22
 10        continue
           sum22=sum22/4.
           return
           end
*****************************************************
*(23)
           function sum23(jj)
           real sum23
           integer ss

           sum23=0
            do 10 ss=23,jj
             sum23=sum23+(ss+22)**2*sum22(ss-1)/10.
10         continue
           sum23=sum23/4.
           return
           end
************************************************
*(24)
```

```
        function sum24(jj)
        real sum24
        integer ss


        sum24=0
         do 10 ss=24,jj
          sum24=sum24+(ss+23)**2*sum23(ss-1)/10.
10      continue
        sum24=sum24/4.
        return
        end
*************************************************
*(25)
        function sum25(jj)
        real sum25
        integer ss
        sum25=0
         do 10 ss=25,jj
          sum25=sum25+(ss+24)**2*sum24(ss-1)/10.
10      continue
        sum25=sum25/4.
        return
        end
*********************************************
        subroutine newton(para,k)
        real para(0:1000),k
        read *,u1,pr
10      u2=u1-f(u1,k,para)/df(u1,k,para)
        error=abs((u2-u1)/u2)
        if(error.gt.pr) then
        u1=u2
        print *,u1,error
        goto 10
        endif
        u2=b*u2
        print *,u2,f(u2/2,k,para)
        return
        end
*********************************
        function f(u,k,para)
        real para(0:1000)
        f=0.
        do 10 i=0,k
         f=f+para(i)*u**(2*i)
10      continue
        return
```

```
          end
****************************************
          function df(u,k,para)
          real para(0:1000)
          df=0.
          do 20 i=1,k+1
           df=df+para(i)*u**(2*i-1)*2*i
20        continue
          return
****************************************
C FUNCTION OF SUMMATION
          function summ(i,j,s,at)
          real at
            if (s .eq. 24) then
                     at=sum24(j)
                     goto 30
            endif
            if (s .eq. 25) then
                     at=sum25(j)
                     goto 30
            endif        if (s .eq. 23 ) then
                     at=sum23(j)
                     goto 30
            endif
            if (s .eq. 22) then
                     at=sum22(j)
                     goto 30
            endif
            if (s .eq. 21) then
                     at=sum21(j)
                     goto 30
            endif
            if (s .eq. 20) then
                     at=sum20(j)
                     goto 30
            endif
            if (s .eq. 19) then
                     at=sum19(j)
                     goto 30
            endif        if (s .eq. 18 ) then
                     at=sum18(j)
                     goto 30
            endif
            if(s .eq. 17) then
                     at=sum17(j)
                     goto 30
```

```
endif
if (s .eq. 16) then
        at=sum16(j)
        goto 30
endif
if (s .eq. 14) then
        at=sum14(j)
        goto 30
endif
if (s .eq. 15) then
        at=sum15(j)
        goto 30
endif      if (s .eq. 13 ) then
        at=sum13(j)
        goto 30
endif
if (s .eq. 12) then
        at=sum12(j)
        goto 30
endif
if (s .eq. 11) then
        at=sum11(j)
        goto 30
endif
if (s .eq. 10) then
        at=sum10(j)
        goto 30
endif
if (s .eq. 9) then
        at=sum9(j)
        goto 30
endif      if (s .eq. 8 ) then
        at=sum8(j)
        goto 30
endif
if(s .eq. 7) then
        at=sum7(j)
        goto 30
endif
if (s .eq. 6) then
        at=sum6(j)
        goto 30
endif
if (s .eq. 4) then
        at=sum4(j)
        goto 30
```

```
            endif
            if (s .eq. 5) then
                        at=sum5(j)
                        goto 30
            endif          if (s .eq. 3 ) then
                        at=sum3(j)
                        goto 30
            endif
            if(s .eq. 2) then
                        at=sum2(j)
                        goto 30
            else
                        at=sum1(j)
            endif
30        returen
          end
*****************************************
```

APPENDIX B

COEFFICIENT d_k OF EIGENFUNCTION FOR DIFFERENT *Kn*

THE COEFFICIENT d_k' OF EIGENFUNCTION FOR DIFFERENT *Kn*

k	0.00	0.02	0.04	0.06	0.08
0	1.000000	1.000000	1.000000	1.000000	1.000000
1	−18.750000	−20.750000	−22.750000	−24.750000	−26.750000
2	79.210100	98.265600	119.321000	142.377000	167.432000
3	−144.043000	−201.296000	−271.689000	−356.553000	−457.224000
4	145.080000	229.024000	344.257000	497.614000	696.594000
5	−92.671500	−165.560000	−277.502000	−442.200000	−676.202000
6	40.861900	82.728000	54.757000	272.024000	454.598000
7	−13.181200	−30.273700	−63.245900	−122.683000	−224.130000
8	3.249410	8.466540	19.757500	42.302900	84.501400
9	−0.646315	−1.883630	−4.887290	−11.531000	−25.168400
10	0.124692	0.363041	1.005790	2.575100	6.103080
11	−3.58121E−02	−7.81416E−02	−0.196144	−0.505606	−1.259180
12	1.48968E−02	2.45660E−02	4.68932E−02	1.02985E−01	0.243432
13	−5.81835E−03	−9.07529E−03	−1.47620E−02	−2.62552E−02	−5.21826E−02
14	1.82761E−03	2.97430E−03	4.78575E−03	7.83189E−03	1.34943E−02
15	−4.58083E−04	−7.98782E−04	−1.34809E−03	−2.23449E−03	−3.71120E−03
16	9.33666E−05	1.75660E−04	3.16460E−04	5.51241E−04	9.39192E−04
17	−1.57875E−05	−3.21078E−05	−6.20691E−05	−1.15009E−04	−2.05969E−04
18	2.25303E−06	4.95574E−06	1.02964E−05	2.03759E−05	3.86913E−05
19	−2.75302E−07	−6.55009E−07	−1.46339E−06	−3.09708E−06	−6.25517E−06
20	2.91544E−08	7.50298E−08	1.80276E−07	4.08225E−07	8.77967E−07
21	−2.70262E−09	−7.52384E−09	−1.94436E−08	−4.71180E−08	−1.07959E−07
22	2.22204E−10	6.68238E−10	1.85579E−09	4.81032E−09	1.17395E−08
23	−1.57586E−11	−5.16599E−11	−1.55104E−10	−4.31820E−10	−1.12584E−09
24	1.17557E−12	4.04916E−12	1.28004E−11	3.75388E−11	1.03039E−10
25	−3.96765E−14	−1.63832E−13	−6.01195E−13	−2.00029E−12	−6.12443E−12

THE COEFFICIENT d_k' OF EIGENFUNCTION FOR DIFFERENT *Kn* (continued)

k	0.005	0.10	0.12
0	1.000000000000	1.000000000000	1.000000000000
1	−19.249999755282	−28.750000000000	−30.750000000000
2	83.786455012016	194.48784722222	223.54340277778
3	−157.19747929320	−575.03255208333	−711.31380208333
4	163.48101833927	949.36357393887	1264.7562007813
5	−107.88202195819	−999.17322697416	−1434.1592495969
6	49.163092053819	728.53075977812	1126.9391090331
7	−16.395674045065	−389.65391363012	−649.68941116624
8	4.1785060393633	159.38657434817	286.48984269996
9	−0.85474908176110	−51.472458492094	−99.748319034885
10	0.16295069004395	13.456265606584	28.116494799992
11	−4.2678849965781D−02	−2.9059588237375	−6.5472351852919
12	1.6696073164466D−02	0.52713284089290	1.2806819641796
13	−6.4933580354236D−03	−8.1451719510272D−02	−0.21339847213796
14	2.0688881265729D−03	1.0849459119075D−02	3.0653718130602D−02
15	−5.2835960492763D−04	−1.2586480084747D−03	−3.8350878281171D−03
16	1.0984968315700D−04	1.2831332514336D−04	4.2164684557941D−04
17	−1.8952355167967D−05	−1.1585667342762D−05	−4.1059428596368D−05
18	2.7599126784909D−06	9.3297867024356D−07	3.5660366873094D−06
19	−3.4412736954670D−07	−6.7424213213213D−08	−2.7794492323325D−07
20	3.7186482821050D−08	4.3971254994017D−09	1.9549974662813D−08
21	−3.5185593897498D−09	−2.6008164814639D−10	−1.2471725946634D−09
22	2.9411030829215D−10	1.4015685304521D−11	7.2489509368086D−11
23	−2.1886373955472D−11	−6.9100732439231D−13	−3.8547073869455D−12
24	1.4598270425440D−12	3.1286794400673D−14	1.8824408023026D−13
25	−8.7801704962411D−14	−1.3054533728857D−15	−8.4718119683390D−15

APPENDIX C

COEFFICIENT $b_{i,j}$ FOR DIFFERENT Kn

I. $b_{i,j}$ for $Kn = 0$ ($\beta = 1$)

j | i = j, j+1, j+2, ... , 2*j–1, 2*j

1 –.2500E+02 0.6250E+01

2 0.1562E+03 –.8681E+02 0.9766E+01

3 –.4340E+03 0.3798E+03 –.9657E+02 0.6782E+01

4 0.6782E+03 –.8138E+03 0.3308E+03 –.5273E+02 0.2649E+01

5 –.6782E+03 0.1036E+04 –.5840E+03 0.1498E+03 –.1709E+02 0.6623E+00

6 0.4710E+03 –.8746E+03 0.6328E+03 –.2265E+03 0.4173E+02 –.3668E+01 0.1150E+00

7 –.2403E+03 0.5256E+03 –.4653E+03 0.2148E+03 –.5541E+02 0.7881E+01 –.5597E+00 0.1467E–01

8 0.9386E+02 –.2364E+03 0.2477E+03 –.1405E+03 0.4692E+02 –.9363E+01 0.1077E+01 –.6381E–01 0.5729E–02

9 –.2897E+02 0.8256E+02 –.1000E+03 0.6740E+02 –.2773E+02 0.7179E+01 –.1160E+01 0.1114E+00 –.2258E–01 0.4420E–03

10 0.7242E+01 –.2304E+02 0.3170E+02 –.2477E+02 0.1213E+02 –.3878E+01 0.8143E+00 –.1100E+00 0.3612E–01 –.1594E–02 0.2763E–04

11 –.1496E+01 0.5258E+01 –.8098E+01 0.7202E+01 –.4100E+01 0.1564E+01 –.4059E+00 0.7132E–01 –.3297E–01 0.2357E–03 –.9193E–04 0.1427E–05

12 0.2598E+00 –.9992E+00 0.1704E+01 –.1700E+01 0.1104E+01 –.4899E+00 0.1520E+00 –.3305E–01 0.1989E–01 –.2002E–02 0.1265E–03 –.4412E–05 0.6194E–07

13 –.3843E–01 0.1606E+00 –.3003E+00 0.3324E+00 –.2425E+00 0.1230E+00 –.4445E–01 0.1156E–01 –.8622E–02 0.1131E–02 –.1006E–03 0.5681E–05 –.1790E–06 0.2291E–08

14 0.4902E–02 –.2211E–01 0.4501E–01 –.5473E–01 0.4436E–01 –.2531E–01 0.1046E–01 –.3176E–02 0.2835E–02 –.4609E–03 0.5345E–04 –.4252E–05 0.2167E–06 –.6216E–08 0.7304E–10

15 –.5446E–03 0.2638E–02 –.5806E–02 0.7696E–02 –.6862E–02 0.4355E–02 –.2028E–02 0.7045E–03 –.7343E–03 0.1430E–03 –.2057E–04 0.2134E–05 –.1532E–06 0.7109E–08 –.1869E–09 0.2029E–11

16 0.5319E–04 –.2753E–03 0.6516E–03 –.9350E–03 0.9099E–03 –.6359E–03 0.3297E–03 –.1291E–03 0.1542E–03 –.3509E–04 0.6051E–05 –.7788E–06 0.7293E–07 –.4767E–08 0.2027E–09 –.4916E–11 0.4953E–13

17 –.4601E–05 0.2535E–04 –.6419E–04 0.9916E–04 –.1046E–03 0.7984E–04 –.4563E–04 0.1991E–04 –.2683E–04 0.7000E–05 –.1411E–05 0.2178E–06 –.2532E–07 0.2158E–08 –.1293E–09 0.5068E–11 –.1140E–12 0.1071E–14

18 0.3550E–06 –.2074E–05 0.5596E–05 –.9260E–05 0.1052E–04 –.8716E–05 0.5446E–05 –.2622E–05 0.3938E–05 –.1160E–05 0.2682E–06 –.4842E–07 0.6753E–08 –.7148E–09 0.5585E–10 –.3084E–11 .1121E–12 –.2348E–14 0.2066E–16

19 –.2459E–07 0.1518E–06 –.4348E–06 0.7674E–06 –.9350E–06
0.8351E–06 –.5665E–06 0.2984E–06 –.4947E–06 0.1626E–06
–.4246E–07 0.8795E–08 –.1435E–08 0.1823E–09 –.1769E–10
0.1274E–11 –.6523E–13 0.2207E–14 –.4326E–16 0.3578E–18

20 0.1537E–08 –.1000E–07 0.3030E–07 –.5681E–07 0.7389E–07
–.7080E–07 0.5183E–07 –.2966E–07 0.5383E–07 –.1954E–07
0.5695E–08 –.1333E–08 0.2498E–09 –.3713E–10 0.4325E–11
–.3871E–12 0.2584E–13 –.1231E–14 0.3896E–16 –.7169E–18
0.5590E–20

21 –.8711E–10 0.5959E–09 –.1905E–08 0.3781E–08 –.5228E–08
0.5351E–08 –.4206E–08 0.2599E–08 –.5127E–08 0.2038E–08
–.6564E–09 0.1716E–09 –.3634E–10 0.6204E–11 –.8461E–12
0.9090E–13 –.7540E–14 0.4687E–15 –.2088E–16 0.6200E–18
–.1075E–19 0.7922E–22

22 0.4499E–11 –.3228E–10 0.1085E–09 –.2274E–09 0.3332E–09
–.3628E–09 0.3047E–09 –.2023E–09 0.4311E–09 –.1864E–09
0.6578E–10 –.1901E–10 0.4497E–11 –.8683E–12 0.1360E–12
–.1711E–13 0.1704E–14 –.1316E–15 0.7650E–17 –.3198E–18
0.8939E–20 –.1464E–21 0.1023E–23

23 –.2126E–12 0.1596E–11 –.5632E–11 0.1242E–10 –.1922E–10
0.2219E–10 –.1984E–10 0.1408E–10 –.3226E–10 0.1508E–10
–.5789E–11 0.1834E–11 –.4797E–12 0.1035E–12 –.1834E–13
0.2651E–14 –.3093E–15 0.2869E–16 –.2071E–17 0.1130E–18
–.4444E–20 0.1173E–21 –.1820E–23 0.1209E–25

24 0.9229E–14 –.7235E–13 0.2673E–12 –.6192E–12 0.1009E–11
–.1231E–11 0.1168E–11 –.8826E–12 0.2163E–11 –.1087E–11
0.4512E–12 –.1556E–12 0.4463E–13 –.1065E–13 0.2109E–14
–.3450E–15 0.4625E–16 –.5026E–17 0.4359E–18 –.2954E–19
0.1516E–20 –.5632E–22 0.1408E–23 –.2073E–25 0.1312E–27

25 –.3692E–15 0.3017E–14 –.1165E–13 0.2827E–13 –.4841E–13
0.6223E–13 –.6241E–13 0.5006E–13 –.1307E–12 0.7028E–13
–.3139E–13 0.1171E–13 –.3656E–14 0.9570E–15 –.2097E–15
0.3835E–16 –.5819E–17 0.7267E–18 –.7385E–19 0.6011E–20
–.3834E–21 0.1858E–22 –.6533E–24 0.1550E–25 –.2172E–27
0.1312E–29

APPENDIX D

THE INTEGRAL OF EQ. (3.18)

The constant C_n in the solution

$$\theta\ (r^*, x^*) = \sum_{n=1}^{\infty} C_n G_n(r^*)\ exp\ [- \frac{2\ (\lambda_n)^2\ x^*\ (1 + 8\ Kn)}{Gz}] \tag{D.1}$$

must satisfy the following condition for all r^*

$$1 = \sum_{n=1}^{\infty} C_n G_n \tag{D.2}$$

To determine C_n, two theorems must first be proved for the functions G_n, like those for Bessel's function.

Theorem I. Let G_i and G_j be two functions satisfying

$$\frac{d^2 G_i}{dr^{*2}} + (\frac{1}{r^*}\frac{dG_i}{dr^*}) + \lambda_i^2 (1 - r^{*2} + 4Kn) G_i = 0 \tag{D.3a}$$

$$\frac{d^2 G_j}{dr^{*2}} + (\frac{1}{r^*}\frac{dG_j}{dr^*}) + \lambda_j^2 (1 - r^{*2} + 4Kn) G_j = 0 \tag{D.3b}$$

if λ_i and λ_j are the roots of equation $G(1) = 0$, and $\lambda_i \neq \lambda_j$, then

$$\int_0^1 G_i G_j\ r^* (1 - r^{*2} + 4Kn)\ dr^* = 0 \tag{D.4}$$

Let λ_i and λ_j be the first two eigenvalues. Multiplying Eq. (D.3a) with $G_j dr^*$, Eq. (D.3b) with $G_i dr^*$ and integrating from 0 to 1, and subtracting each other, we obtain

$$(\lambda_i^2 - \lambda_j^2) \int_0^1 G_i G_j\ r^* (1 - r^{*2} + 4Kn)\ dr^* = (G_i \frac{dG_j}{dr^*} - G_j \frac{dG_i}{dr^*})_{r^*=1} \tag{D.5}$$

Because both $G_i\ (1) = 0$ and $G_j\ (1) = 0$, $\lambda_i \neq \lambda_j$, therefore, Eq. (D.4) must be true.

Theorem II. The value of the integral in Eq. (D.4) for $i = j$ can be determined by

$$\int_0^1 G_i G_i \, r^* (1-r^{*2} + 4Kn) \, dr^* = \frac{1}{2\lambda_i} \left(\frac{dG_i}{d\lambda_i} \frac{dG_i}{dr^*} \right)_{r^*=1} \tag{D.6}$$

Let λ_i and λ_j be the first two eigenvalues again, and the Eq. (D.5) is true. Now let

$$\lambda_i = \lambda_i + d\lambda_i \quad , \qquad G_j = G_i + \frac{dG_i}{d\lambda_i} d\lambda_i$$

and

$$2\lambda_i \int_0^1 G_i G_i \, r^* (1-r^{*2} + 4Kn) \, dr^* = \left(\frac{\partial G_i}{\partial \lambda_i} \frac{\partial G_i}{\partial r^*} - G_i \frac{\partial^2 G_i}{\partial \lambda_i \, \partial r^*} \right)_{r^*=1} \tag{D.7}$$

Since the special λ_i is one of the roots of equation $G(1) = 0$, we have

$$\int_0^1 G_i G_i \, r^* (1-r^{*2} + 4Kn) \, dr^* = \frac{1}{2\lambda_i} \left(\frac{\partial G_i}{\partial \lambda_i} \frac{\partial G_i}{\partial r^*} \right)_{r^*=1} \tag{D.8}$$

With the help of these two theorems, multiplying the Eq. (D.2)

$$1 = \sum_{n=1}^{\infty} C_n G_n$$

with $G_i \, r^*(1- r^{*2}) \, dr^*$ and integrating, we obtain

$$\int_0^1 G_i \, r^* (1-r^{*2} + 4Kn) \, dr^* = C_i \int_0^1 G_i G_j \, r^* (1-r^{*2} + 4Kn) \, dr^* \tag{D.9}$$

From the differential equation (Eq. (D.3a)), the left side of the integration equals to

$$-\frac{1}{\lambda_i^2} \left(\frac{dG_i}{dr^*} \right)_{r^*=1}$$

Therefore,

$$C_i = -\frac{2}{\lambda_i} \frac{1}{\left(\frac{dG_i}{d\lambda_i} \right)_{r^*=1}} \tag{D.10}$$

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